Name: ____________________ Material: A-Math Mock Paper 2021

OVERMUGGED MOCK PAPER 2021
SECONDARY 4 EXPRESS
SECONDARY 5 NORMAL ACADEMIC

ADDITIONAL MATHEMATICS **4049/02**

Specimen Paper MARKING SCHEME

Date: **1 September 2021** **Duration: 2 hours 15 minutes**

Candidates answer on separate writing paper

READ THESE INSTRUCTIONS FIRST

Write your name on all work you hand in.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer all questions.
If working is needed for any question it must be shown with the answer.
Omission of essential working will result in loss of marks.
You are expected to use a scientific calculator to evaluate explicit numerical expressions.
If the degree of accuracy is not specified in the question, and if the answer is not exact, give the answer to three significant figures.
Give answers in degrees to one decimal place.
For π, use either your calculator value of 3.142, unless the question requires the answer in terms of π.

At the end of the exam, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question
The total number of marks for this paper is 90.

Setter: Ong Kai Wen

This question paper consists of 20 printed pages including the cover page

MATHEMATICAL FORMULAE

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

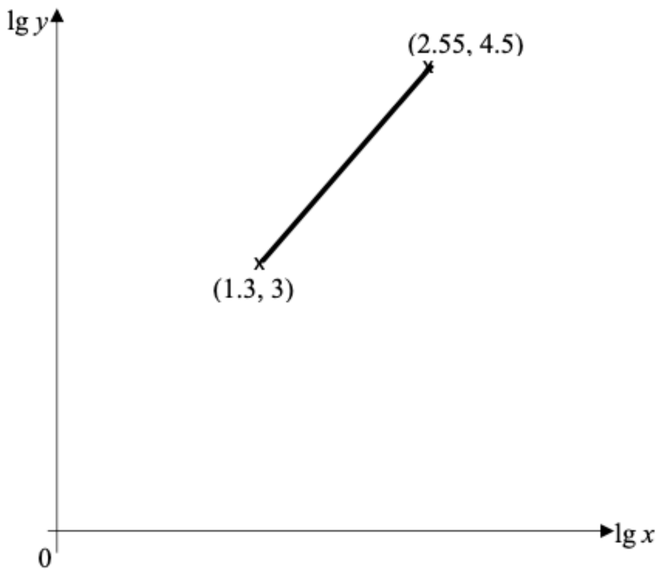
Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$\Delta = \frac{1}{2}bc \sin A$$

1.



The variables x and y are such that when $\lg y$ is plotted against $\lg x$, the straight line graph shown below is obtained

(a) Find the value of A and of b when

$$y = Ax^b$$

[5]

Solution

$y = Ax^b$

$\lg y = b \lg x + \lg A$

$Y = mX + c$

$\text{Gradient} = b$

$= \frac{4.5 - 3}{2.55 - 1.3}$

$= 1\frac{1}{5}$

Substituting $(1.3, 3)$ into the equation,

$\lg y = b \lg x + \lg A$

$\lg A = 3 - \left(1\frac{1}{5}\right)(1.3)$

$= 1.44$

$A = 10^{1.44}$

$= 27.542287 \ldots$

$= 27.5 \ (3.\text{s.f.})$

(b) Find the value of $\lg y$ when $x = 100$

[2]

Solution

$$\lg y = \left(1\frac{1}{5}\right)\lg(100) + 1.44$$
$$= 3.84$$

(c) Find the value of x when $y = 8000$

[2]

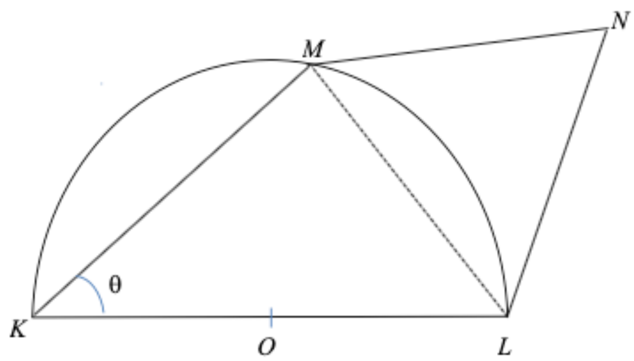
Solution

$$\lg y = \left(1\frac{1}{5}\right)\lg x + 1.44$$
$$\lg(8000) = \left(1\frac{1}{5}\right)\lg x + 1.44$$
$$\lg x = \frac{\lg(8000) - 1.44}{\left(1\frac{1}{5}\right)}$$
$$x = 10^{\frac{\lg(8000)-1.44}{\left(1\frac{1}{5}\right)}}$$
$$= 112.869081 \ldots$$
$$= 113\ (3.\text{s.f.})$$

[S4 HYSS P2/2020 PRELIM Qn 4]

[**Total: 9 marks**]

2.



The diagram shows a triangle $\boldsymbol{KLM}$ is a semi-circle with centre $\boldsymbol{O}$ and radius $\mathbf{2\ cm}$. $\boldsymbol{M}$ is a point on the circumference such that $\angle \boldsymbol{MKL} = \boldsymbol{\theta}°$. An equilateral triangle $\boldsymbol{LNM}$ is drawn on side $\boldsymbol{LM}$

(a) The area of the quadrilateral $\boldsymbol{KLNM}$ is $\boldsymbol{A}\ \mathbf{cm^2}$. Show that

$$A = 4\sin 2\theta - 2\sqrt{3}\cos 2\theta + 2\sqrt{3}$$

[5]

<u>Solution</u>

$ML = 4\sin\theta$

$KM = 4\cos\theta$

$$\text{Area of } \Delta KML = \frac{1}{2}(4\sin\theta)(4\cos\theta)$$
$$= 4\sin 2\theta$$

$$\text{Area of } \Delta MNL = \frac{1}{2}(4\sin\theta)^2 \sin 60°$$
$$= (8\sin^2\theta)\left(\frac{\sqrt{3}}{2}\right)$$
$$= 4\sqrt{3}\sin^2\theta$$
$$= 4\sqrt{3}\left(\frac{1-\cos 2\theta}{2}\right)$$
$$= 2\sqrt{3} - 2\sqrt{3}\cos 2\theta$$

$$\text{Total area} = 4\sin 2\theta - 2\sqrt{3}\cos 2\theta + 2\sqrt{3}\ \text{(shown)}$$

(b) Find the value of R and of α when $4\sin 2\theta - 2\sqrt{3}\cos 2\theta$ is expressed as $R\sin(2\theta - \alpha)$, where R and α are constant

[2]

Solution

$$4\sin 2\theta - 2\sqrt{3}\cos 2\theta = R\sin(2\theta - \alpha)$$

$$R = \sqrt{4^2 + \left(2\sqrt{3}\right)^2}$$
$$= \sqrt{28}$$

$$\alpha = \tan^{-1}\left(\frac{2\sqrt{3}}{4}\right)$$
$$= 40.89339\ldots$$
$$= 40.9^\circ \text{ (1.d.p.)}$$

(c) Hence, find the maximum area of the quadrilateral $KLNM$ and the corresponding value of θ and the corresponding perimeter

[5]

Solution

$$A = \sqrt{28}\sin(2\theta - 40.9°) + 2\sqrt{3}$$

$$\text{Maximum } A = \sqrt{28} + 2\sqrt{3}$$
$$= 8.755604 \ldots$$
$$= 8.76 \text{ cm}^2 \text{ (3.s.f.)}$$

$$\text{Max } A \text{ occurs when } \sin(2\theta - \alpha) = 1$$

$$2\theta - \tan^{-1}\left(\frac{2\sqrt{3}}{4}\right) = 90°$$

$$\theta = \frac{90 + \tan^{-1}\left(\frac{2\sqrt{3}}{4}\right)}{2}$$
$$= 65.446697 \ldots$$
$$= 65.4° \text{ (1.d.p.)}$$

$$\text{Perimeter of } KLNM = 4\cos\left(\frac{90 + \tan^{-1}\left(\frac{2\sqrt{3}}{4}\right)}{2}\right) + 8\sin\left(\frac{90 + \tan^{-1}\left(\frac{2\sqrt{3}}{4}\right)}{2}\right) + 4$$
$$= 12.937859 \ldots$$
$$= 12.9 \text{ cm (3.s.f.)}$$

[S4 HSS P2/2020 PRELIM Qn 11]

[Total: 12 marks]

3.

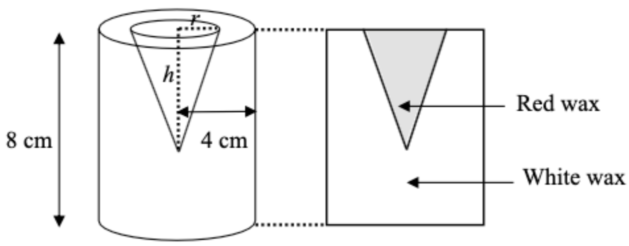


Diagram A Diagram B (cross-sectional view)

Diagram $\boldsymbol{A}$ shows a cylindrical candle stick of height $\mathbf{8\ cm}$ and radius $\mathbf{4\ cm}$. The candle stick is made of white wax and red wax as shown in Diagram $\boldsymbol{B}$. The red wax forms an inverted cone of radius $\boldsymbol{r}$ **cm** and height $\boldsymbol{h}$ **cm**

(a) Given that the sum of the radius and height of the cone is to remain constant at $\mathbf{5\ cm}$, express $\boldsymbol{h}$ in terms of $\boldsymbol{r}$

[1]

Solution

$$h = 5 - r$$

(b) Show that the volume, V, of the white wax is given by

$$V = 128\pi - \frac{5}{3}\pi r^2 + \frac{1}{3}\pi r^3$$

[3]

Solution

$$V = \pi(r)^2(h) - \frac{1}{3}\pi r^2 h$$
$$= \pi(4)^2(8) - \frac{1}{3}\pi(r)^2(5 - r)$$
$$= 128\pi - \frac{5}{3}\pi r^2 + \frac{1}{3}\pi r^3 \text{ (shown)}$$

(c) Find the value of $\boldsymbol{r}$ for which $\boldsymbol{V}$ is stationary

[4]

Solution

$$\frac{dV}{dr} = -\frac{10}{3}\pi r + \pi r^2$$

At stationary point, $\frac{dV}{dr} = 0$

$$-\frac{10}{3}\pi r + \pi r^2 = 0$$
$$\pi r\left(-\frac{10}{3} + r\right) = 0$$
$$r = 0 \text{ (rej)} \quad \text{or} \quad r = 3\frac{1}{3}$$

[S4 HIHS P2/2020 PRELIM Qn 2]

[Total: 8 marks]

4. **(a)** Solve the equation

$$\log_3(x-2) = \log_3(12-x) - 2$$

[4]

Solution

$\log_3(x-2) = \log_3(12-x) - 2$

$\log_3(12-x) - \log_3(x-2) = 2$

$\log_3\left(\frac{12-x}{x-2}\right) = 2$

$\frac{12-x}{x-2} = 3^2$

$12 - x = 9x - 18$

$x = 3$

(b) Express y in terms of x given that

$$(2\log_5 y)(\log_x 5) = 8$$

[3]

Solution

$(2\log_5 y)(\log_x 5) = 8$

$2\left(\frac{\log_x y}{\log_x 5}\right)(\log_x 5) = 8$

$\log_x y = 4$

$y = x^4$

(c) Explain why there are no real solutions for the following equation

$$e^{4x} + 7 = 3e^{2x}$$

[Hint: Substitution]

[3]

Solution

$e^{4x} + 7 = 3e^{2x}$

Let $u = e^{2x}$,

$u^2 - 3u + 7 = 0$

$b^2 - 4ac = (-3)^2 - 4(1)(7)$

$= -19 < 0$

Since the discriminant is less than 0, there are no real solutions for the equation

[S4 CCHS(M) P2/2020 PRELIM Qn 2]

[Total: 10 marks]

5. A circle, C_1, has equation

$$x^2 + y^2 - 4x + 6y = 36$$

(a) Find the radius and the coordinates of the centre of C_1

[3]

Solution

$\text{Centre} = (2, -3)$

$$\begin{aligned}\text{Radius} &= \sqrt{(2)^2 + (-3)^2 - (-36)} \\ &= \sqrt{49} \\ &= 7 \text{ units (rej} - \text{ve)}\end{aligned}$$

The highest point on a second circle, C_2, is $(3, 6)$ and the equation of the tangent at the lowest point is $y = 4$

(b) Find the radius and the coordinates of the centre of C_2

[2]

Solution

Since the highest point is (3, 6) and the lowest point has a y coordinate of 4,
Radius = 1 unit

$\text{Centre} = (3, 5)$

(c) Find the equation of C_2

[1]

Solution

$C_2: (x - 3)^2 + (y - 5)^2 = 1$

(d) Showing your working clearly, determine whether the circles C_1 and C_2 meet each other

[2]

Solution

$$\begin{aligned}\text{Length of centre from } C_1 \text{ and } C_2 &= \sqrt{(3-2)^2 + \left(5 - (-3)\right)^2} \\ &= \sqrt{65} > (7 + 1)\end{aligned}$$

Since the length between C_1 and C_2 is longer than the sum of the radii combined, the 2 circles will intersect each other

[S4 CHIJ SNGS P2/2020 PRELIM Qn 3]

[**Total: 8 marks**]

6. The following curve, where a is a constant, has two stationary points

$$y = \frac{a}{2x+1} + 9x - 5$$

(a) Find the value of y given that one of the stationary points is

$$\left(-\frac{1}{6}, y\right)$$

[5]

Solution

$$y = \frac{a}{2x+1} + 9x - 5$$

$$\frac{dy}{dx} = -\frac{2a}{(2x+1)^2} + 9$$

Since there are 2 stationary points, $\left.\frac{dy}{dx}\right|_{x=-\frac{1}{6}} = 0$

$$0 = -\frac{2a}{\left(2\left(-\frac{1}{6}\right)+1\right)^2} + 9$$

$$2a = 4$$

$$a = 2$$

Substitute the point into the curve,

$$y = \frac{2}{2\left(-\frac{1}{6}\right)+1} + 9\left(-\frac{1}{6}\right) - 5$$

$$= -3\frac{1}{2}$$

(b) Find the coordinates of the other stationary point

[3]

Solution

To find the other stationary points, $\frac{dy}{dx} = 0$

$$0 = -\frac{4}{(2x+1)^2} + 9$$

$$\frac{4}{(2x+1)^2} = 9$$

$$(2x+1)^2 = \frac{4}{9}$$

$$2x + 1 = \pm\frac{2}{3}$$

$$x = -\frac{5}{6}$$

$$y = \frac{2}{2\left(-\frac{5}{6}\right)+1} + 9\left(-\frac{5}{6}\right) - 5$$

$$= -15\frac{1}{2}$$

Coordinate $= \left(-\frac{5}{6}, -15\frac{1}{2}\right)$

(c) Determine the nature of these stationary points

[3]

Solution

$$\frac{d^2y}{dx^2} = \frac{16}{(2x+1)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{1}{6}} = \frac{16}{\left(2\left(-\frac{1}{6}\right)+1\right)^3}$$
$$= 54 > 0$$

$$x = -\frac{1}{6} \text{ is a minimum point}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{5}{6}} = \frac{16}{\left(2\left(-\frac{5}{6}\right)+1\right)^3}$$
$$= -54 < 0$$

$$x = -\frac{5}{6} \text{ is a maximum point}$$

[S4 CHIJ CHIJS P2/2020 PRELIM Qn 12]

[**Total: 11 marks**]

7. **(a)** Prove that

$$\frac{\operatorname{cosec}^2\theta - 2}{\operatorname{cosec}^2\theta} = \cos 2\theta$$

[3]

Solution

$$\text{LHS} = \frac{\operatorname{cosec}^2\theta - 2}{\operatorname{cosec}^2\theta}$$
$$= 1 - \frac{2}{\operatorname{cosec}^2\theta}$$
$$= 1 - 2\sin^2\theta$$
$$= \cos 2\theta$$
$$= \text{RHS(shown)}$$

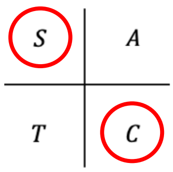
(b) Hence, solve the equation, for $0 < \theta < 5$

$$\frac{\operatorname{cosec}^2\theta - 2}{\operatorname{cosec}^2\theta} + \frac{3}{\operatorname{cosec} 2\theta} = 0$$

[4]

Solution

$$\frac{\operatorname{cosec}^2\theta - 2}{\operatorname{cosec}^2\theta} + \frac{3}{\operatorname{cosec} 2\theta} = 0$$
$$\cos 2\theta + 3\sin 2\theta = 0$$
$$3\sin 2\theta = -\cos 2\theta$$
$$\tan 2\theta = -\frac{1}{3}$$
$$\alpha = \tan^{-1}\left(\frac{1}{3}\right)$$



$$2\theta = \pi - \tan^{-1}\left(\frac{1}{3}\right), 2\pi - \tan^{-1}\left(\frac{1}{3}\right), 3\pi - \tan^{-1}\left(\frac{1}{3}\right)$$
$$\theta = \frac{\pi - \tan^{-1}\left(\frac{1}{3}\right)}{2}, \frac{2\pi - \tan^{-1}\left(\frac{1}{3}\right)}{2}, \frac{3\pi - \tan^{-1}\left(\frac{1}{3}\right)}{2}$$
$$= 1.41\ (3.\text{s.f.}) \quad = 2.98\ (3.\text{s.f.}) \quad = 4.55\ (3.\text{s.f.})$$

[S4 BPGHS P2/2020 PRELIM Qn 6]

[Total: 7 marks]

8. A particle $\boldsymbol{P}$ leaves a fixed point $\boldsymbol{O}$ and moves in a straight line so that, $\boldsymbol{t}$ **seconds** after leaving $\boldsymbol{O}$, its velocity, $\boldsymbol{V}$ **m/s** is given by the following, where $\boldsymbol{k}$ is a constant

$$v = -3t^2 + kt + 15$$

(a) Given that its deceleration is $8\ \mathrm{m/s^2}$ when $t = 2$, find the value of k

[2]

Solution

$a = -6t + k$

When $t = 2$,

$-8 = -6(2) + k$

$k = 4$

(b) Find the value of t for which the particle is at instantaneous rest

[2]

Solution

At instantaneous rest, $v = 0$

$-3t^2 + 4t + 15 = 0$

$3t^2 - 4t - 15 = 0$

$(t - 3)(3t + 5) = 0$

$t = 3 \quad \text{or} \quad t = -\frac{5}{3}\ (\text{rej})$

(c) Find the maximum velocity of the particle

[2]

Solution

At maximum velocity, $\frac{dv}{dt} = 0$

$-6t + 4 = 0$

$t = \frac{2}{3}$

When $t = \frac{2}{3}$,

$V = -3\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right) + 15$

$= 16\frac{1}{3}\ \mathrm{m/s}$

(d) Find the time taken for the particle to return to point $\boldsymbol{O}$

[3]

Solution

$$s = \int -3t^2 + 4t + 15\,dt$$
$$= -t^3 + 2t^2 + 15t + c$$

When $t = 0, s = 0$

$c = 0$

$$s = -t^3 + 2t^2 + 15t$$

When $s = 0$,

$-t^3 + 2t^2 + 15t = 0$

$-t(t^2 - 2t - 15) = 0$

$-t(t + 3)(t - 5) = 0$

$t = 0\ (\text{rej}) \quad \text{or} \quad t = 3\ (\text{rej}) \quad \text{or} \quad t = 5$

[S4 BGSS P2/2020 PRELIM Qn 11]

[**Total: 9 marks**]

9. **(a)** Without using a calculator, show that

$$\tan 15° = 2 - \sqrt{3}$$

[4]

Solution

$$\begin{aligned}\tan 15° &= \tan(45° - 30°)\\ &= \frac{\tan 45° - \tan 30°}{1 + (\tan 45°)(\tan 30°)}\\ &= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} \times \frac{1 - \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}\\ &= \frac{1 - \frac{2}{\sqrt{3}} + \frac{1}{3}}{\left(\frac{2}{3}\right)}\\ &= \left(\frac{4}{3} - \frac{2}{\sqrt{3}}\right) \div \frac{2}{3}\\ &= \left(\frac{4\sqrt{3} - 6}{3\sqrt{3}}\right)\left(\frac{3}{2}\right)\\ &= \frac{2\sqrt{3} - 3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\\ &= 2 - \sqrt{3} \text{ (shown)}\end{aligned}$$

(b) Hence, find the values of the integers a and b such that

$$\sec^2 15° = a - b\sqrt{3}$$

[3]

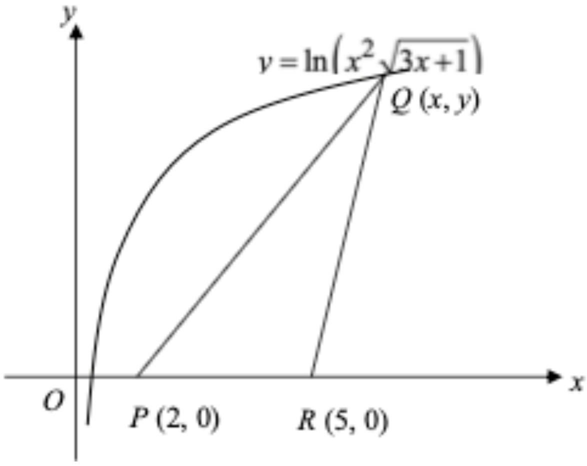
Solution

$$\begin{aligned}\sec^2 15° &= 1 + \tan^2 15°\\ &= 1 + \left(2 - \sqrt{3}\right)^2\\ &= 1 + 4 - 4\sqrt{3} + 3\\ &= 8 - 4\sqrt{3}\end{aligned}$$

[S4 ACS(B) P2/2020 PRELIM Qn 9(a)]

[**Total: 7 marks**]

10.



The diagram shows the curve and three points $P(2, 0)$, $Q(x, y)$ and $R(5, 0)$. The point $Q(x, y)$ lies on the curve

$$y = \ln\left(x^2\sqrt{3x+1}\right)$$

(a) Show that the area, A units^2, of the triangle PQR is given by

$$A = 3\ln x + \frac{3}{4}\ln(3x+1)$$

[2]

Solution

$$A = \frac{1}{2}(PR)(y)$$

$$= \frac{1}{2}(3)\ln\left(x^2\sqrt{3x+1}\right)$$

$$= \frac{3}{2}\ln\left(x^2\sqrt{3x+1}\right)$$

$$= \frac{3}{2}\left(\ln x^2 + \ln(3x+1)^{\frac{1}{2}}\right)$$

$$= \frac{3}{2}\left(2\ln x + \frac{1}{2}\ln(3x+1)\right)$$

$$= 3\ln x + \frac{3}{4}\ln(3x+1) \text{ (shown)}$$

(b) Given that x is increasing at a rate of $\mathbf{0.2\ units/s}$, find the rate at which the area, A, is changing at the instant when $x = \mathbf{15\ units}$

[3]

Solution

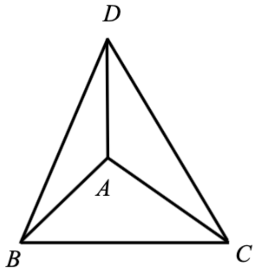
$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$
$$= \left[\frac{3}{x} + \frac{3}{4}\left(\frac{3}{3x+1}\right)\right](0.2)$$
$$= \frac{3}{5x} + \frac{9}{20(3x+1)}$$

$$\left.\frac{dA}{dt}\right|_{x=15} = \frac{3}{5(15)} + \frac{9}{20(3(15)+1)}$$
$$= 0.0497826\ldots$$
$$= 0.0498\ \text{units}^2/\text{s}$$

[S4 AHS P1/2020 PRELIM Qn 9]

[**Total**: **5 marks**]

11.



A solid triangular pyramid, $\boldsymbol{ABCD}$, with base $\boldsymbol{ABC}$ and vertex $\boldsymbol{D}$ such that $\boldsymbol{D}$ is vertically above $\boldsymbol{A}$, has a base area of $\left(\mathbf{8}+\mathbf{2}\sqrt{\mathbf{5}}\right)\,\mathbf{cm^2}$ and height $\left(\mathbf{12}-\sqrt{\mathbf{5}}\right)\,\mathbf{cm}$. The top part of the pyramid is removed by a cut parallel to its base and passing through the midpoint of $\boldsymbol{AD}$. Find the volume of the remaining solid, leaving your answer in the following form, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are integers

$$\frac{7\left(a+b\sqrt{5}\right)}{12}\ \text{cm}^3$$

[4]

Solution

$$\begin{aligned}\text{Volume of solid remaining} &= \left(1-\left(\frac{1}{2}\right)^3\right)\left(\frac{1}{3}\right)\left(8+2\sqrt{5}\right)\left(12-\sqrt{5}\right)\\ &= \left(\frac{7}{8}\right)\left(\frac{1}{3}\right)\left(8+2\sqrt{5}\right)\left(12-\sqrt{5}\right)\\ &= \frac{7\left(96-10+16\sqrt{5}\right)}{24}\\ &= \frac{7\left(43+8\sqrt{5}\right)}{12}\ \text{cm}^3\end{aligned}$$

[S4 ANDSS P1/2020 PRELIM Qn 10(b)]

[**Total**: **4 marks**]

End of Paper ☺