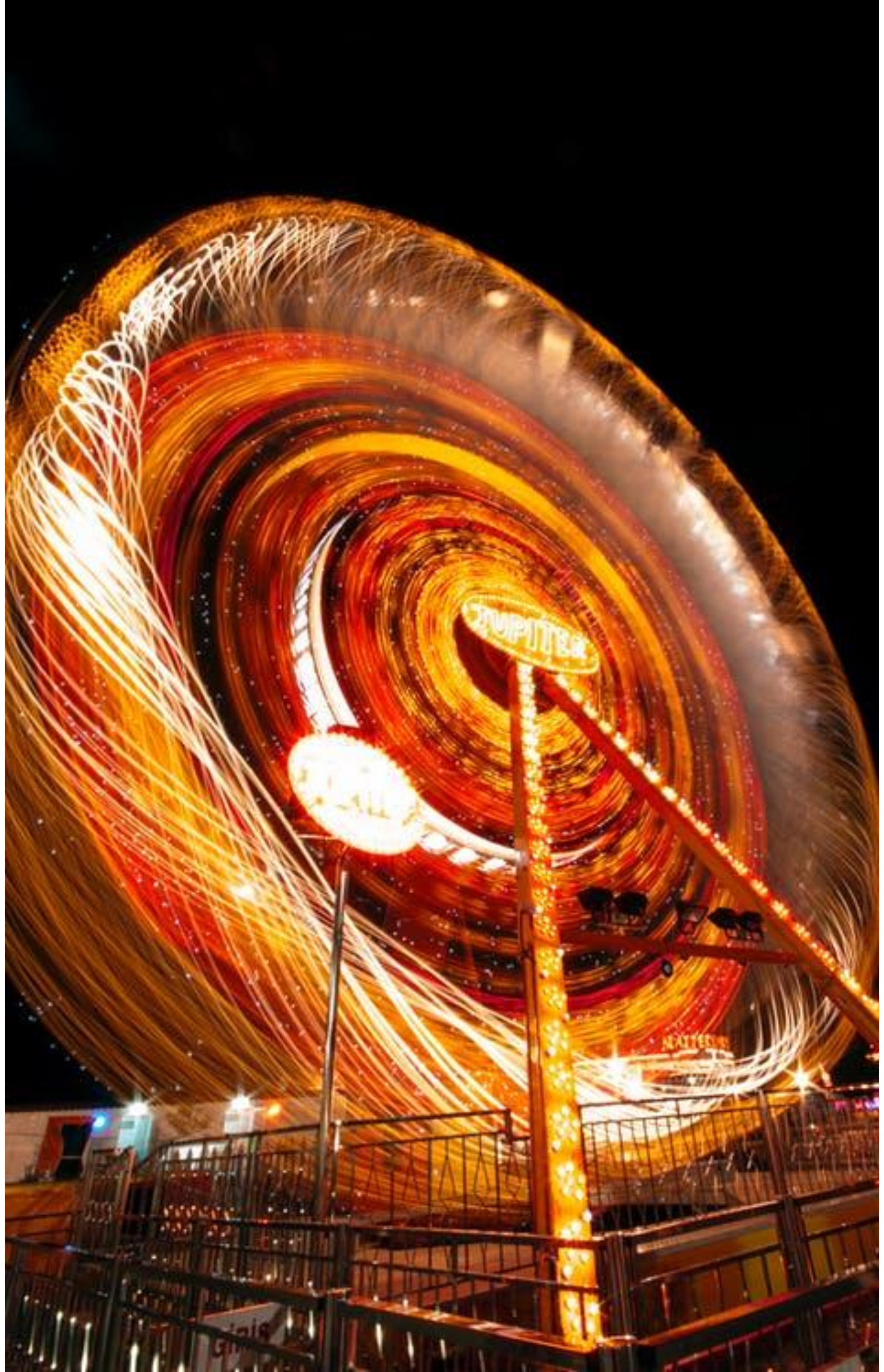
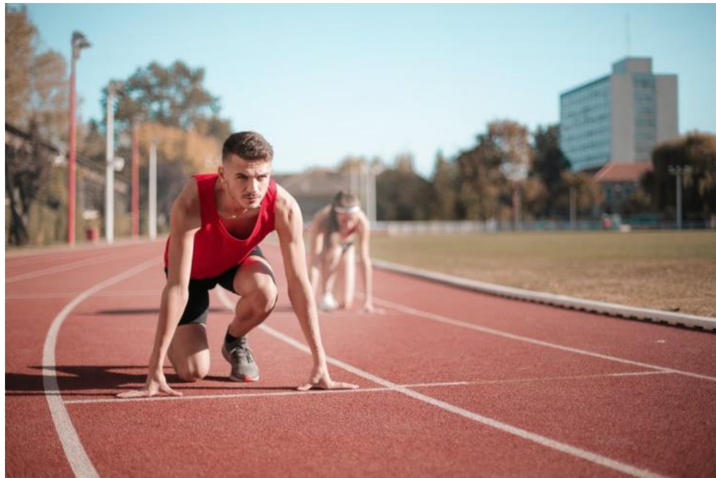
Motion in a Circle

Overmugged





Content

- Kinematics of Uniform Circular Motion
- Dynamics of Uniform Circular Motion
- Non-uniform Circular Motion

The Kinematics of Circular Motion

- Circular motion is a common occurrence in our daily lives: spinning discs, Ferris wheels, and rotation of a tyre about its axis. In this topic, we explore the physics behind rotating objects.

Angular Displacement

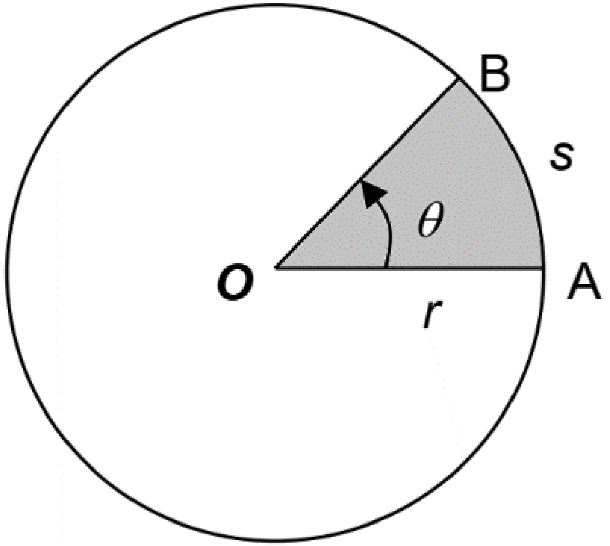
Symbol: θ
SI unit: radians [rad]
Vector quantity

- The angle through which an object turns, usually measured in radians (rad)

Consider an object going round a circular path of radius *r* from point A to B.

$$\theta = \frac{s}{r}$$

where $s =$ arc length
$r =$ radius of circle



Radians

- Is the angle subtended by an arc length equal to the radius of the arc

$$\pi = 180°$$

Conversion from radians to degrees:

$$x\,rad = x\left(\frac{180°}{\pi}\right)$$

Conversion from degrees to radians:

$$y\,° = y\left(\frac{\pi}{180}\right)\,rad$$

Angular Velocity

Symbol: ω

SI unit: radians per second [rad/s]

Vector

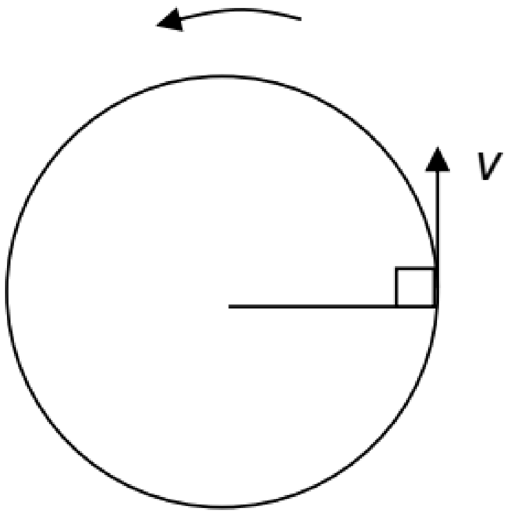
- is the rate of change of angular displacement with respect to time.
- Its direction is either *clockwise* or *anticlockwise*

$$\omega = \frac{d\theta}{dt}$$

where $\theta =$ angular displacement

$t =$ time

Relationship between angular velocity ω and linear velocity v



By differentiating $\theta = \frac{s}{r}$ with respect to *t*

$$\frac{d\theta}{dt} = \frac{1}{r}\left(\frac{ds}{dt}\right) \text{ where } r \text{ is constant}$$

$$\omega = \frac{1}{r}(v)$$

Hence,

$$v = r\omega$$

where v = linear or tangential velocity
r = radius of circular path
ω = angular velocity

- v has units of m/s.
- The direction of the linear velocity *v* of a body at a particular point along its circular motion is along the tangent drawn to the circular path at that point and therefore always perpendicular to the radius.

Period

Symbol: *T*
SI unit: seconds [s]

- The period of an object in circular motion is the time taken for it to make one complete revolution.

Frequency

Symbol: *f*
SI unit: Hertz [Hz]

- The frequency of an object in circular motion is the number of complete revolutions made per unit time.

$$f = \frac{1}{T}$$

Relationship between period, angular velocity, frequency

For an object moving in uniform circular motion, i.e. with constant ω

$$\omega = \frac{2\pi}{T}$$

Since $T = 1/f$

$$\omega = 2\pi f$$

Relationship between period and linear speed

If the object is moving in uniform circular motion, its linear speed v can also be found by:

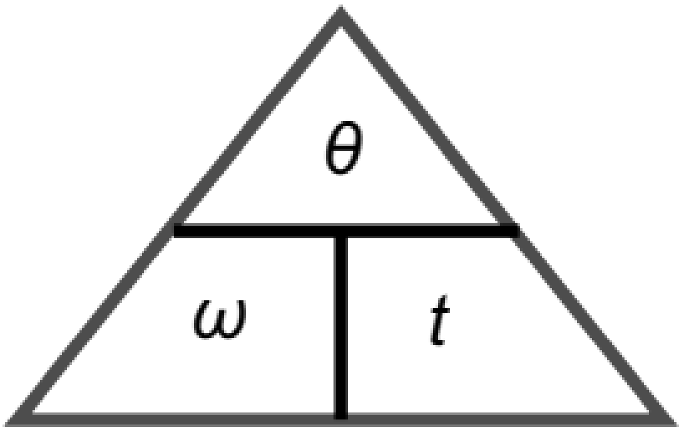
$$v = \frac{circumference\ of\ the\ circle}{T}$$

Thus

$$T = \frac{2\pi r}{v}$$

In summary

For uniform circular motion, it may help to adopt a primary school approach



Practice Example 1

Compact disc players use laser to scan the content of a disc. Suppose the laser is pointed 3.5 cm from the center of the disc, what length of the disc is scanned when the disc rotated by an angle of 30°.

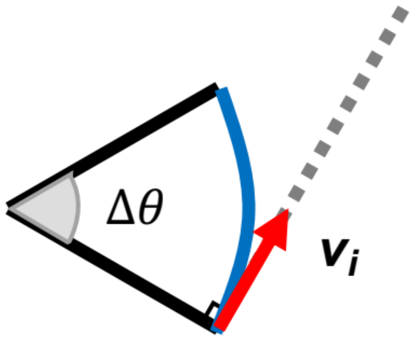
Practice Example 2

As part of Kenneth's weekly workout, he jogs in a circular track of radius 250 m near his apartment building. He completes one full round in 10 minutes. Assuming that he jogs at constant angular speed,

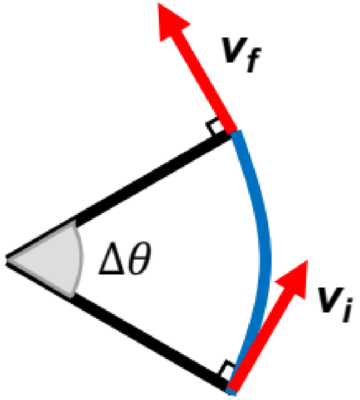
a) What is the distance covered in one full round?
b) What is Kenneth's angular speed?
c) Calculate its linear speed?
d) Find the frequency of its motion.

Dynamics of Uniform Circular Motion

- Consider an object moving in uniform circular motion with a speed of v and an angular velocity of ω.

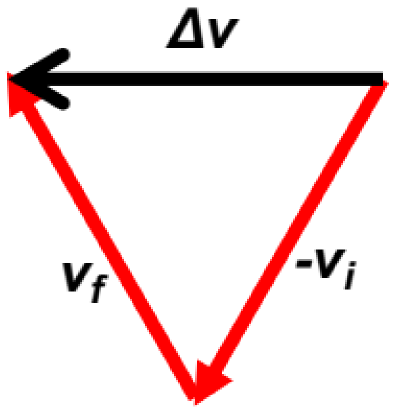


- By Newton's 1st Law, if there is no net force on an object, it will continue in its state of constant velocity. Inertia will cause the object to continue moving along a straight line.

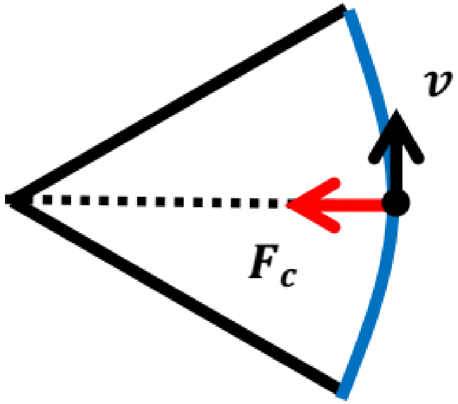


- For an object undergoing uniform circular motion, the speed remains constant, $|\boldsymbol{v_i}| = |\boldsymbol{v_f}|$, but the velocity changes due to direction changing, $\boldsymbol{v_f} \neq \boldsymbol{v_i}$. Thus, there must be an acceleration.

Dynamics of Uniform Circular Motion



- The change in velocity of the object is given by $\Delta \boldsymbol{v} = \boldsymbol{v}_f - \boldsymbol{v}_i$. This implies a change in momentum $\Delta \boldsymbol{p} = \Delta(\boldsymbol{mv})$. By Newton's 2nd Law, there is a corresponding resultant force, $\boldsymbol{F} \propto \frac{\boldsymbol{dp}}{\boldsymbol{dt}}$, acting along the same direction as the change in momentum



- The net force and correspondingly the acceleration are directed towards the center of the circular path. ***Centripetal*** means center-seeking

Centripetal Acceleration

- Is the acceleration which acts perpendicular to the velocity and always acts towards the center of the circular motion.

$$a_c = v\omega = r\omega^2 = \frac{v^2}{r}$$

where a_c = centripetal acceleration
r = radius of circular motion
v = linear or tangential velocity
ω = angular velocity

Centripetal Force

- The force that causes an object to go round the circle

$$F_c = mr\omega^2 = mv\omega = \frac{mv^2}{r}$$

where F_c = centripetal force
r = radius of circular motion
v = linear or tangential velocity
ω = angular velocity
m = mass of the object

- The centripetal force F_C is a resultant force. Do not include it in the free body diagram.
- The centripetal force F_C does no work. The displacement is perpendicular to the force.

General Approach in Solving Problems relating to Circular Motion

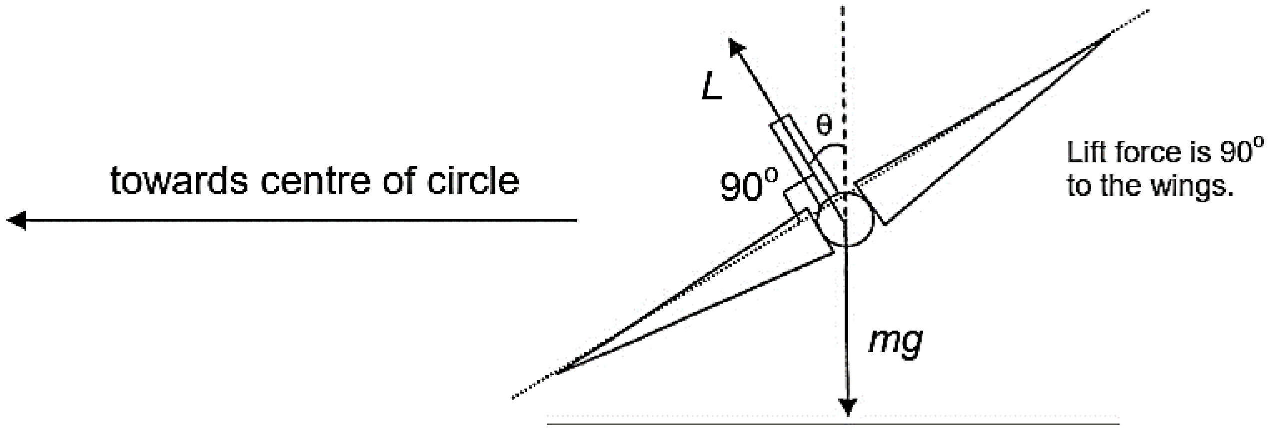
1) Draw a free body diagram showing all the forces acting on the body.
2) Determine the center of the circular path.
3) Resolve the forces in the direction towards the center of the circle.
4) Identify the force(s) that provide the centripetal force and write down the statement. e.g. **Friction provides the centripetal force**
5) Apply Newton's Second Law:

$$F_{net} = mr\omega^2 = mv\omega = \frac{mv^2}{r}$$

Worked Example 1

Airplane making a circular turn

When an airplane makes a turn, it has to tilt the wings at angle to the vertical. This allows a component of the lift force *L* to provide for the centripetal force.



If the plane flies in a horizontal circle (i.e. vertical forces are balanced), then

$$L\cos\theta = mg$$

Horizontal component of lift provides the centripetal force:

$$L\sin\theta = \frac{mv^2}{r}$$

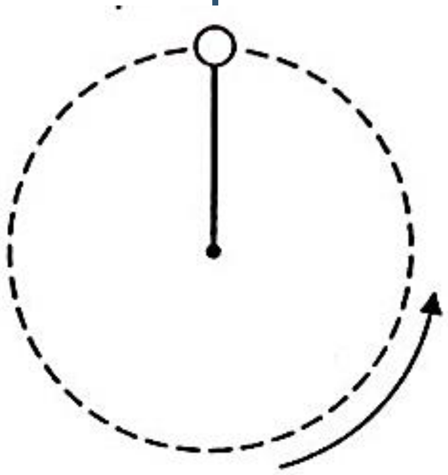
Dividing the second by the first equation

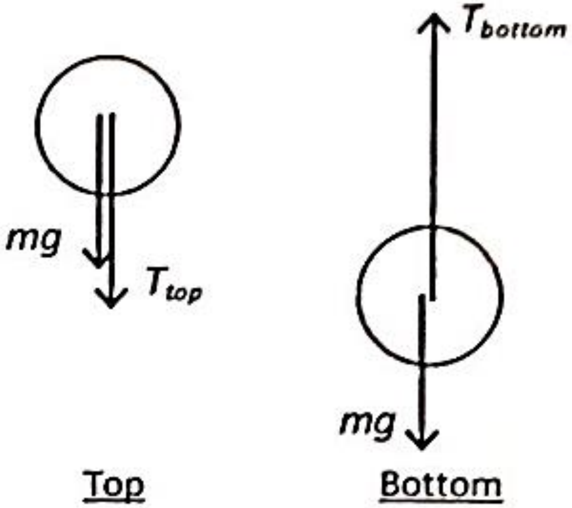
$$\tan\theta = \frac{v^2}{rg}$$

Worked Example 2

A ball moving in a vertical circle at uniform speed

A ball of mass m is attached to a light rod and is moving in a vertical circle of radius r at uniform speed v.





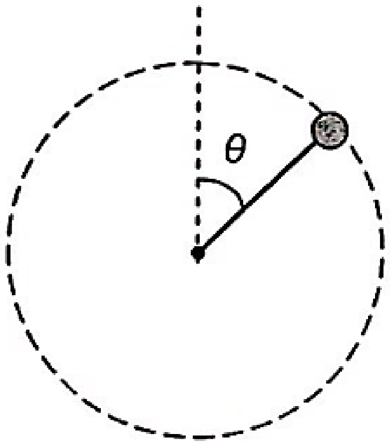
At the top,

$$T_{top} + mg = \frac{mv^2}{r} \Rightarrow T_{top} = \frac{mv^2}{r} - mg$$

At the bottom,

$$T_{bottom} - mg = \frac{mv^2}{r} \Rightarrow T_{bottom} = \frac{mv^2}{r} + mg$$

Note that the tension in the rod is changing according to θ.



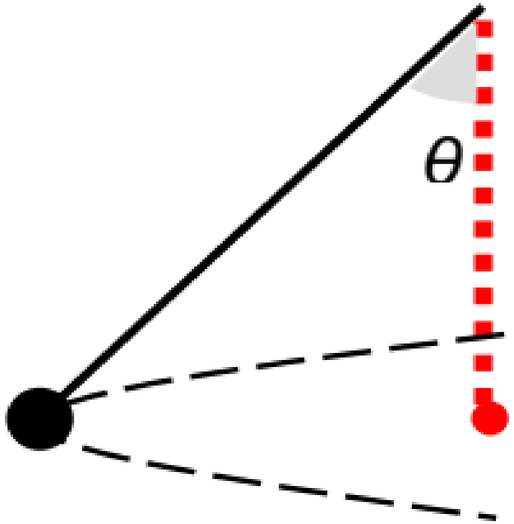
$$T + mg\cos\theta = \frac{mv^2}{r}$$

$$T = \frac{mv^2}{r} - mg\cos\theta$$

Practice Example 3

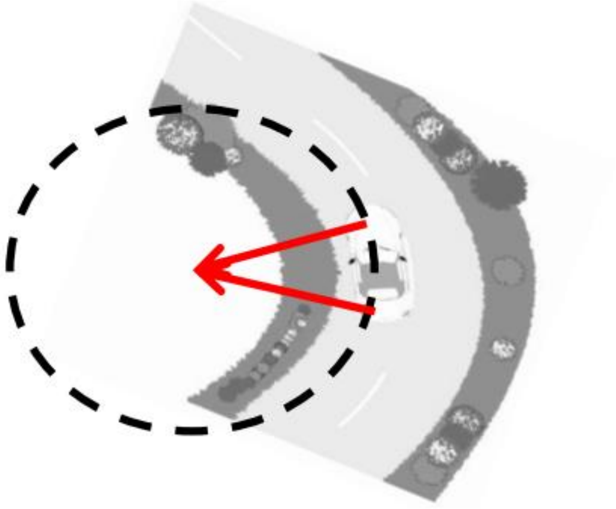
A pendulum bob is suspended by a string and made to undergo uniform circular motion in a fixed horizontal plane.

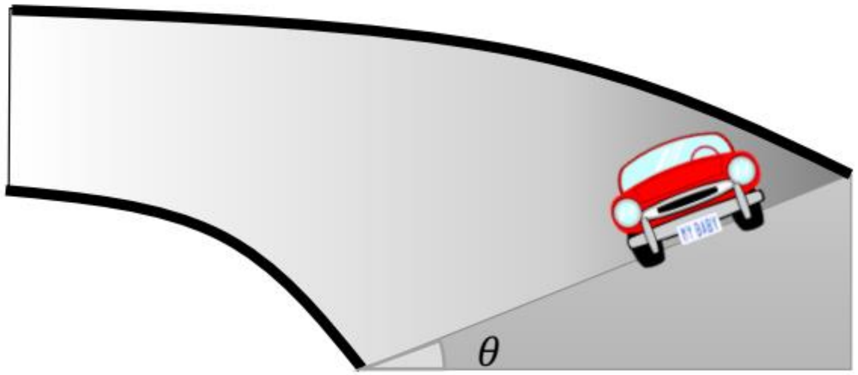
Given that the weight of the bob is 3.50 N, the radius of the circular motion 7.00 cm, and the angle made by the string from the vertical is 11.0°, find the period of circular motion.



Practice Example 4

A car is driven round an icy road banked θ at an angle of 25° and a radius r of 50 m as shown.





a) Calculate the maximum speed that the car can travel at while maintaining the circular radius by assuming that the frictional force between the tyres and the road is negligible.

b) The angle of the banked road is increased to 90°, such that the car is moving on the inside of a vertical cylinder.

   i. Draw the free body diagram of the car.

   ii. Which force provides the centripetal force?

Non-Uniform Circular Motion

- A body moving along a circular path with a speed that is varying is said to be in **non-uniform circular motion.**

Non-uniform circular Motion in a Vertical Circle

- When the circle is vertical, the effects of gravity can cause the object to go slower at the top of the circle than at the bottom. Such situations can be examined by analyzing the energy transformations that take place and/or applying Newton’s laws of motion.
- The acceleration of this body comprises two components: *centripetal acceleration* (or radial acceleration) a_r and acceleration in the *tangential direction* a_t.

- The **centripetal acceleration** of the body keeps it in its circular path, and **can be calculated using the same formula.**

- The tangential acceleration of the body causes its speed to change.

Non-Uniform Circular Motion

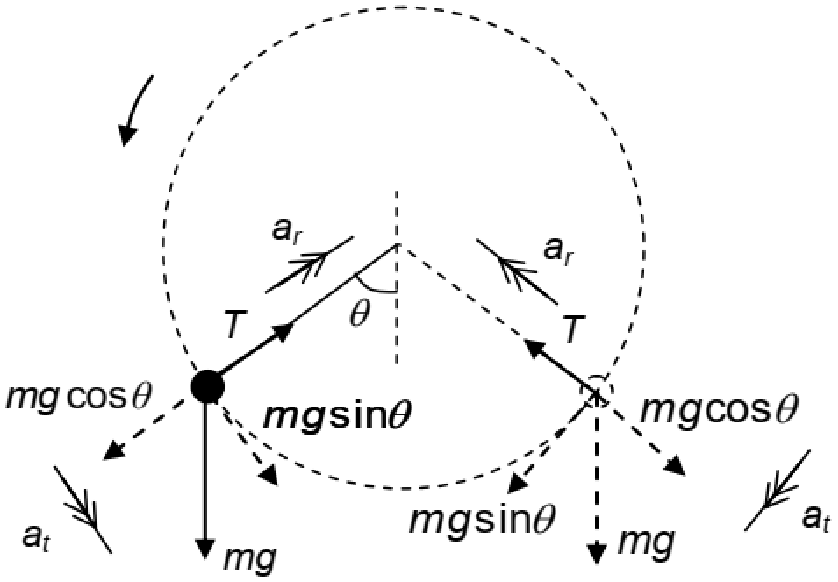
- A body moving along a circular path with a speed that is varying is said to be in **non-uniform circular motion.**

Non-uniform circular Motion in a Vertical Circle

- When the circle is vertical, the effects of gravity can cause the object to go slower at the top of the circle than at the bottom. Such situations can be examined by analyzing the energy transformations that take place and/or applying Newton's laws of motion.
- The acceleration of this body comprises two components: *centripetal acceleration* (or radial acceleration) a_r and acceleration in the *tangential direction* a_t.
- The **centripetal acceleration** of the body keeps it in its circular path, and **can be calculated using the same formula.**
- The tangential acceleration of the body causes its speed to change.

Non-uniform circular Motion in a Vertical Circle

- An object is fixed to one end of a string which rotates in a vertical circle at constant speed.



Consider the free body diagram of the object when the rod makes an angle θ with the vertical.

- The component of the resultant force towards the center of the circle provides the centripetal force. This is also known as the radial force:

$$T - mg\cos\theta = \frac{mv^2}{r}$$

$$\Rightarrow T = \frac{mv^2}{r} + mg\cos\theta$$

At the bottom ($\theta = 0$ and 2π rad), the tension is the greatest:

$$T = \frac{mv^2}{r} + mg$$

At the bottom ($\theta = \pi$ rad), the tension is the smallest:

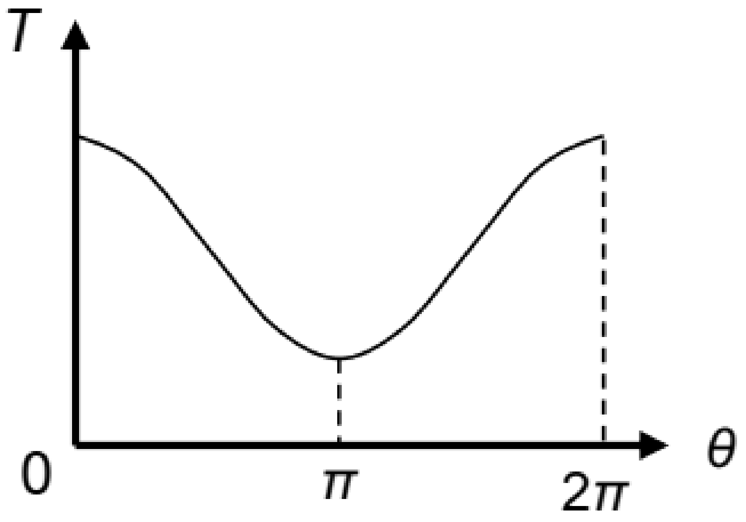
$$T = \frac{mv^2}{r} - mg$$

Non-uniform circular Motion in a Vertical Circle

At the side,

$$T = \frac{mv^2}{r}$$

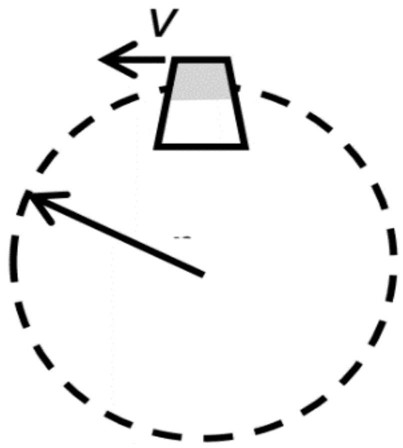
A graph that represents the variation with angle θ of the tension T in the rod:



The tangential resultant force $mg \sin\theta$ causes the speed to increase as the object moves towards the bottom. As it moves away from the bottom, this force slows down the object.

Practice Example 5

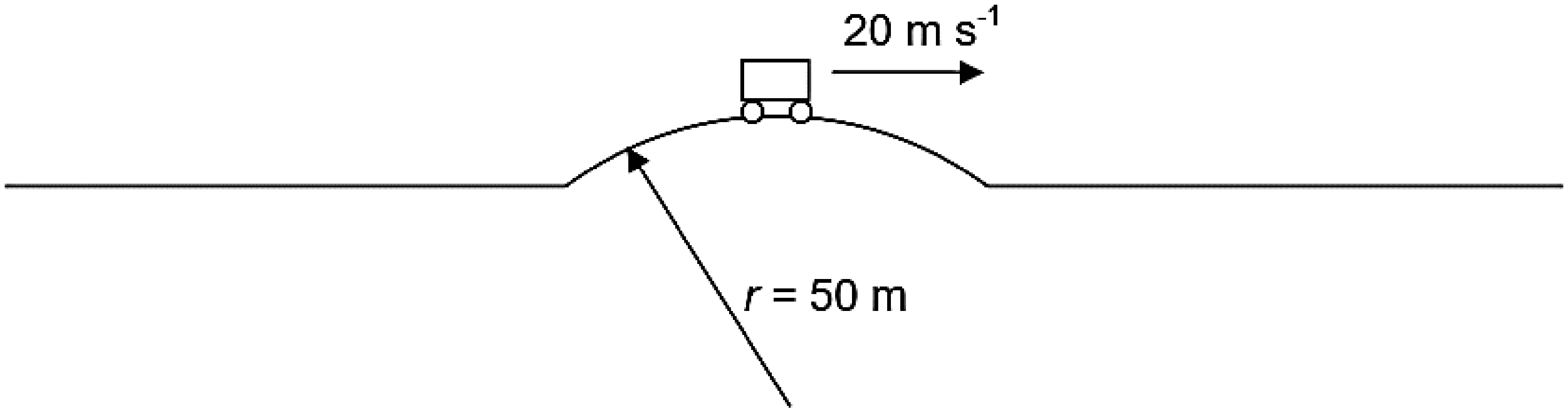
A bucket of water is spun in a circular motion along a vertical plane. Find the minimum linear speed to ensure that the water stays in the bucket.



Practice Example 6

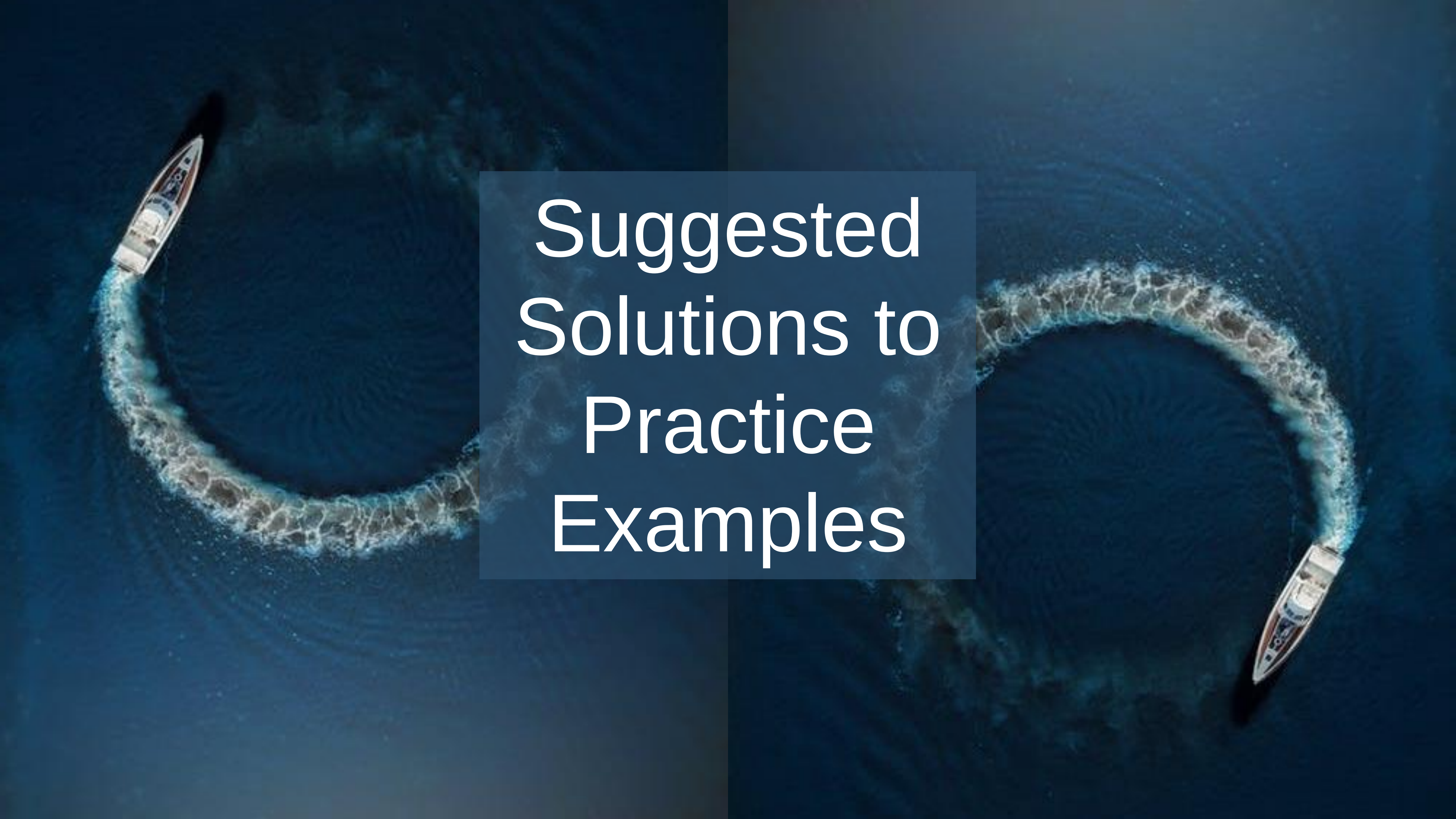
At the top of the bridge, the speed of a 1500 kg car is 20 m/s.

(a) Draw all the forces acting on the car while it is at the top of the bridge.
(b) Calculate the force the car exerts on the road while it is at the top of the bridge.
(c) Calculate the maximum speed of the car for it to remain in contact with the top of the hump.



Summary

Concept	Definition	Formula or important concepts
Angular Displacement	The angle through which an object turns, usually measured in radians (rad). A more explicit definition is: The angular displacement is defined as the ratio of the arc length to its radius.	$\theta = \frac{s}{r}$
Angular Velocity	Angular velocity is defined as the rate of change of displacement.	In general, $\omega = \frac{d\theta}{dt}$ For uniform circular motion, $\omega = \frac{2\pi}{T} = 2\pi f$
Relationship between Angular Displacement and Velocity		$v = r\omega$
Centripetal Acceleration		$a = r\omega^2 = v\omega = \frac{v^2}{r}$
Centripetal Force		$F = ma = mr\omega^2 = mv\omega = \frac{mv^2}{r}$



Suggested Solutions to Practice Examples

Practice Example 1

Compact disc players use laser to scan the content of a disc. Suppose the laser is pointed 3.5 cm from the center of the disc, what length of the disc is scanned when the disc rotated by an angle of 30°.

Solution:

Note that $30° = 30\left(\frac{\pi}{180}\right) = 0.52\, radians$

$$s = r\theta = 3.5(0.52)$$

$$s = 1.82 \text{ cm}$$

Practice Example 2

As part of Kenneth's weekly workout, he jogs in a circular track of radius 250 m near his apartment building. He completes one full round in 10 minutes. Assuming that he jogs at constant angular speed,

a) What is the distance covered in one full round?
b) What is Kenneth's angular speed?
c) Calculate its linear speed?
d) Find the frequency of its motion.

Solution:

(a) Distance, s

$$s = r\theta = r(2\pi) = 250(2\pi) = 1570\, m$$

(b) Angular speed θ

$$\omega = \frac{\theta}{t} = \frac{2\pi}{10\, min}\left(\frac{1\, min}{60\, s}\right) = 10.5 \times 10^{-3}\, rad/s$$

(c) Linear speed v

$$v = r\omega = (250)(10.5 \times 10^{-3}) = 2.6\, m/s$$

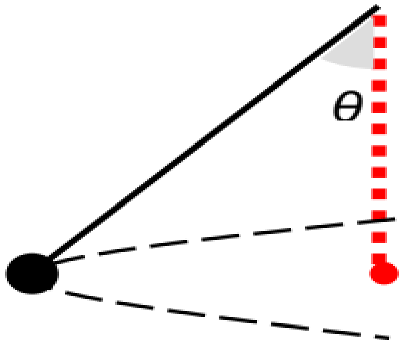
(d) Frequency *f*

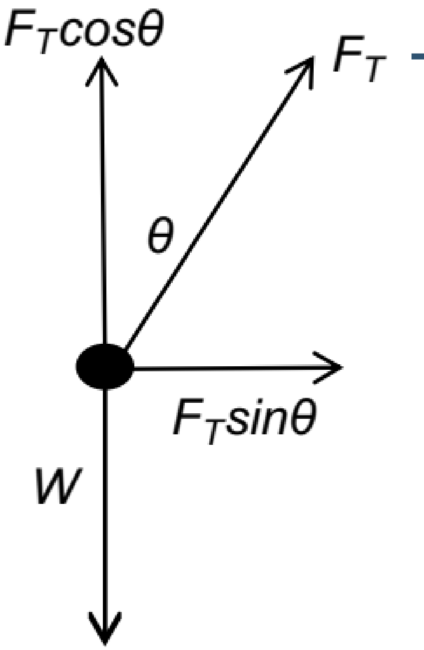
$$f = \frac{1}{T} = \frac{1}{10min}\left(\frac{1min}{60\, s}\right) = 1.67 \times 10^{-3}\, Hz$$

Practice Example 3

A pendulum bob is suspended by a string and made to undergo uniform circular motion in a fixed horizontal plane.

Given that the weight of the bob is 3.50 N, the radius of the circular motion 7.00 cm, and the angle made by the string from the vertical is 11.0°, find the period of circular motion.





Solution:

We resolve all real forces parallel and perpendicular to the plane of the circle

As the object is in equilibrium in the y direction,

$$W = F_T \cos\theta$$

As the x component of the tension provides the centripetal force,

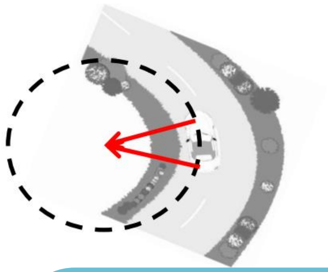
$$F_c = F_T \sin\theta = mr\omega^2$$

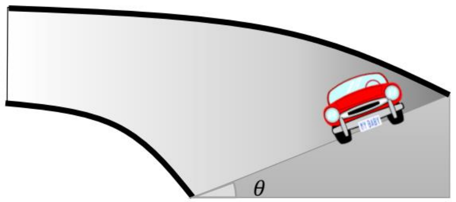
$$\frac{mr\omega^2}{W} = \tan\theta = \frac{mr\omega^2}{mg}$$

$$\left(\frac{2\pi}{T}\right) = \sqrt{\frac{g}{r}\tan\theta}$$

$$T = \frac{2\pi}{\sqrt{\frac{g}{r}\tan\theta}} = \frac{2\pi}{\sqrt{\frac{981}{0.07}\tan 11}} = 1.20\ s$$

Practice Example 4



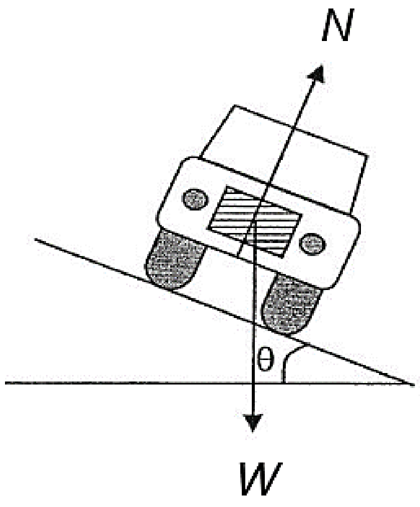


A car is driven round an icy road banked θ at an angle of 25° and a radius r of 50 m as shown.

a) Calculate the maximum speed that the car can travel at while maintaining the circular radius by assuming that the frictional force between the tyres and the road is negligible.

Solution:

Horizontal component of Normal contact force provides centripetal force



$$F_{net} = ma_c$$

$$N\sin\theta = \frac{mv^2}{r}$$

For verticle equilibrium:

$$N\cos\theta = mg$$

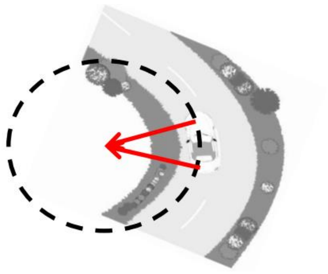
$$\tan\theta = \frac{v^2}{rg}$$

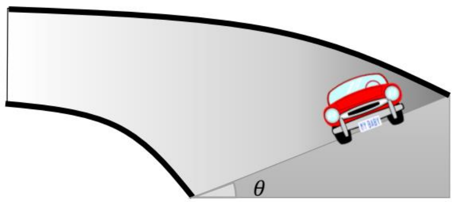
$$v = \sqrt{(50)(9{,}81)(\tan 25)}$$

$$v = 15.1\ m/s$$

Note that when there is friction, horizontal component of friction also contributes to centripetal force: the equation in the second line becomes $N\sin\theta + (friction)\cos\theta = \frac{mv^2}{r}$

Practice Example 4



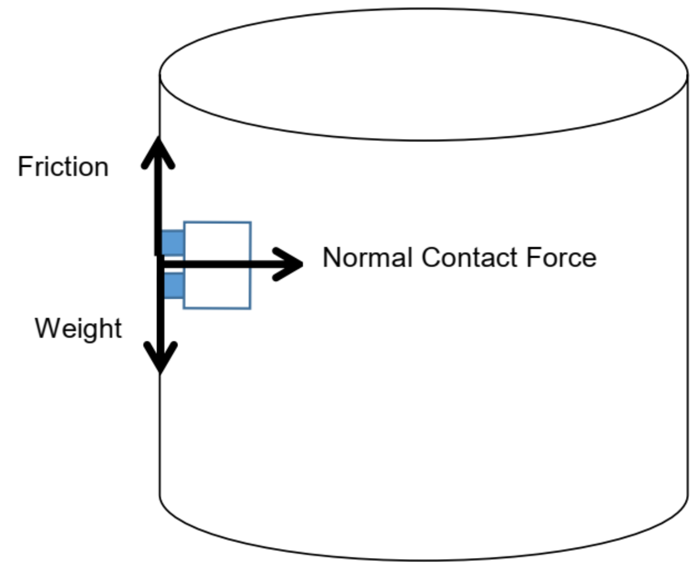


b) The angle of the banked road is increased to 90°, such that the car is moving on the inside of a vertical cylinder.

i. Draw the free body diagram of the car.

ii. Which force provides the centripetal force?

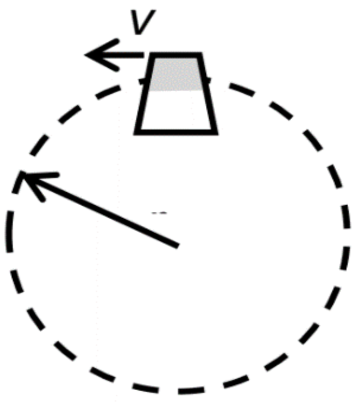
(i)

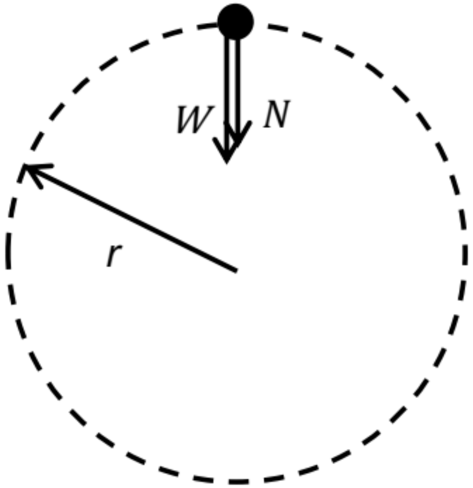


(ii) Normal contact force provides the centripetal force

Practice Example 5

A bucket of water is spun in a circular motion along a vertical plane. Find the minimum linear speed to ensure that the water stays in the bucket.





Solution:

From the free body diagram, we realise that the normal reaction force by the bucket on the water, and the weight of the water provides the centripetal acceleration. However, if the bucket is to just reach the maximum point, the water must be travelling fast enough such that its weight alone can support the circular motion.

$$F_c = W$$

If it travels faster, the centripetal force must be larger which may be provided by the reaction force by bucket on the water.

$$F_C = N + W$$

Therefore,

$$F_C \geq W$$

$$\frac{mv^2}{r} \geq mg$$

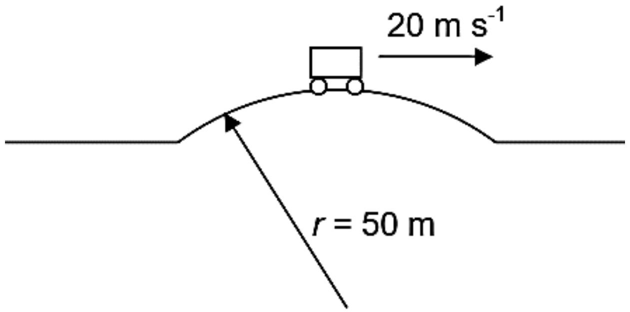
$$v \geq \sqrt{rg}$$

If the bucket travels too slowly, $F_C \leq W$ the centripetal force is too small. Since *N* is negative, there is no contact force and the water spills.

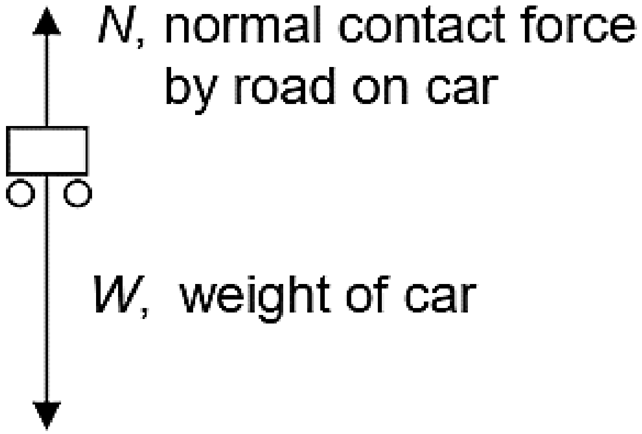
Practice Example 6

At the top of the bridge, the speed of a 1500 kg car is 20 m/s.

(a) Draw all the forces acting on the car while it is at the top of the bridge.
(b) Calculate the force the car exerts on the road while it is at the top of the bridge.
(c) Calculate the maximum speed of the car for it to remain in contact with the top of the hump.



a) Free body diagram



b) The resultant force provides the centripetal force

$$F_{net} = \frac{mv^2}{r} \Rightarrow W - N = \frac{mv^2}{r}$$

$$N = mg - \frac{mv^2}{r} = 1500\left(9.81 - \frac{20^2}{50}\right) = 2720\ N$$

By Newton's 3rd law, force by the car on the road is also 2720 N.

c) Maximum speed

$$W - N = \frac{mv^2}{r} \Rightarrow W - \frac{mv^2}{r} = N$$

to remain in contact, $N > 0$

$$mg - \frac{mv^2}{r} > 0$$

$$v < \sqrt{rg}$$

$$v < \sqrt{(50)(9.81)} = 22.1\ m/s$$

Hence the maximum speed is 22.1 m/s