KINEMATICS

Overmugged

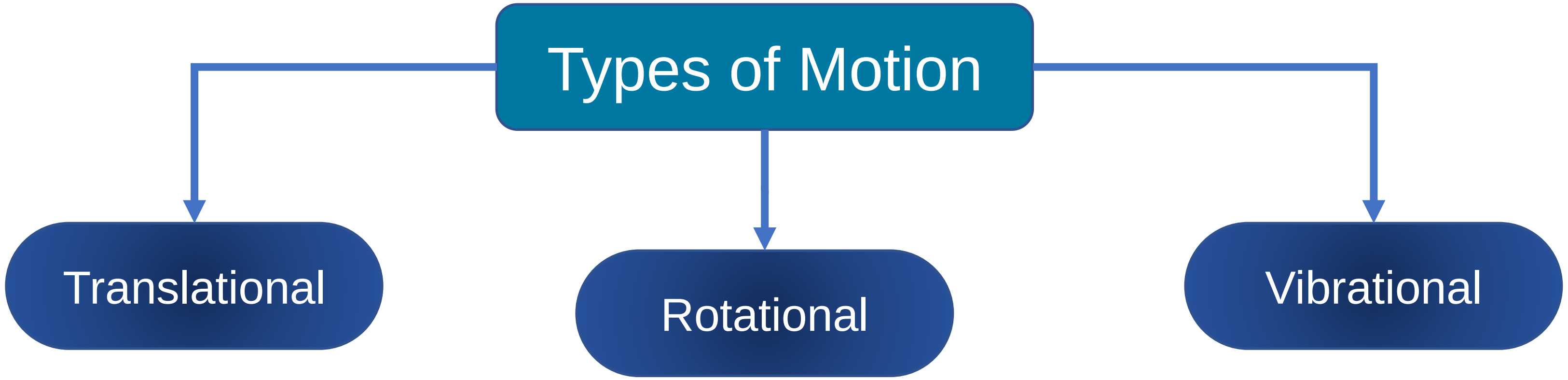
Contents

Mechanics

- The study of motion of objects, with associated concepts of force and energy.

Two Branches

1. Kinematics
    - describe motion of objects without regard of forces
2. Dynamics
    - describe motion that result from forces



Rectilinear Motion

- Motion in one dimension

Symbol: d
SI unit: meter (m)
Scalar

Distance

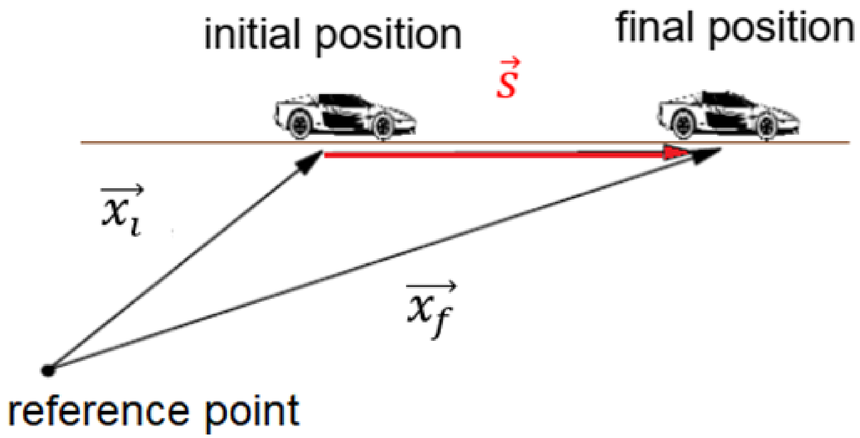
- the total length of path covered by a moving object

Symbol: $\vec{s}$
SI unit: meter (m)
Vector

Displacement

- The linear distance of the position of the moving object from a given reference point.

$$\vec{s} = \overrightarrow{x_f} - \overrightarrow{x_i}$$



Positive displacement

- Motion to the right (or upwards)

Negative displacement

- Motion to the left (or downwards)

Rectilinear Motion

Symbol: v_{ave}
SI unit: m/s
Scalar

Average Speed

- total distance travelled over the total time taken.

$$v_{ave} = \frac{d}{\Delta t}$$

$d =$ total distance travelled
$\Delta t =$ time

Instantaneous Speed

- rate of change of distance over time.

Symbol: $\vec{v}$
SI unit: m/s
Vector

Average Velocity

- total displacement travelled over the total time taken

$$\vec{v}_{ave} = \frac{\vec{s}}{\Delta t}$$

Instantaneous velocity

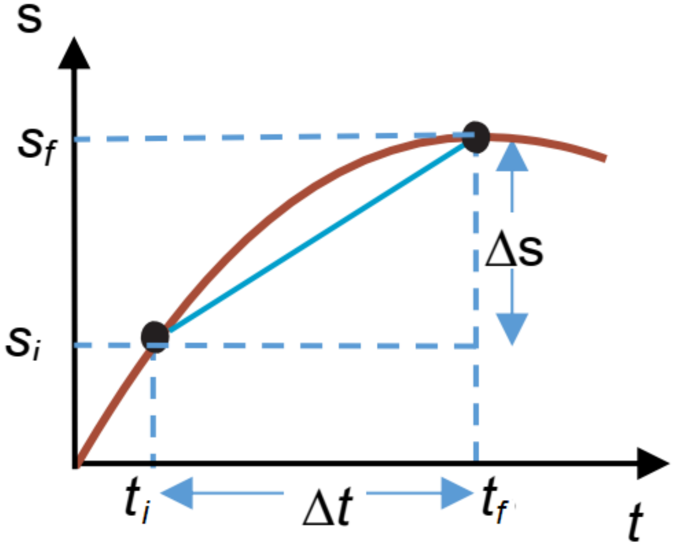
- rate of change of displacement over time.

$$\vec{v} = \frac{d\vec{s}}{dt}$$

Rectilinear Motion

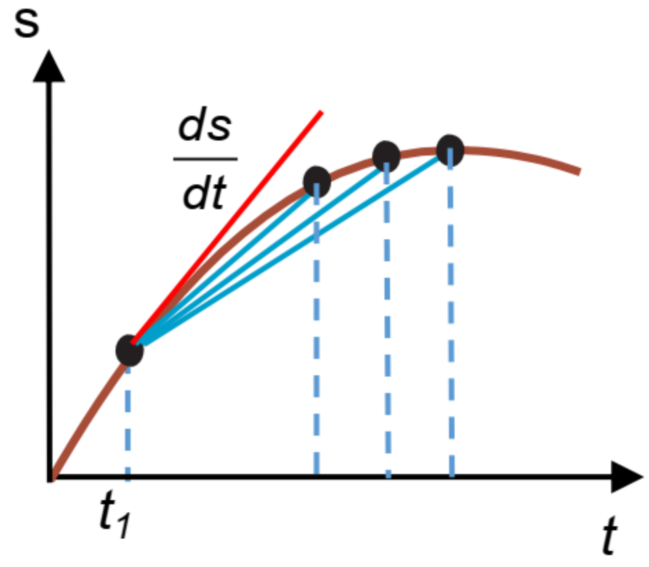
Average Velocity

$$\vec{v}_{ave} = \frac{\vec{s}}{\Delta t}$$



Instantaneous velocity

$$\vec{v} = \frac{d\vec{s}}{dt}$$



Instantaneous velocity at a point in time t is the slope of the tangent line at that point in x vs t plot.

NOTE: 1. The area under the curve of an v vs t graph will give the change in displacement.

$\int v \, dt = \Delta s$

2. A **positive velocity** indicates motion towards **right**; **negative** for motion to the **left.**

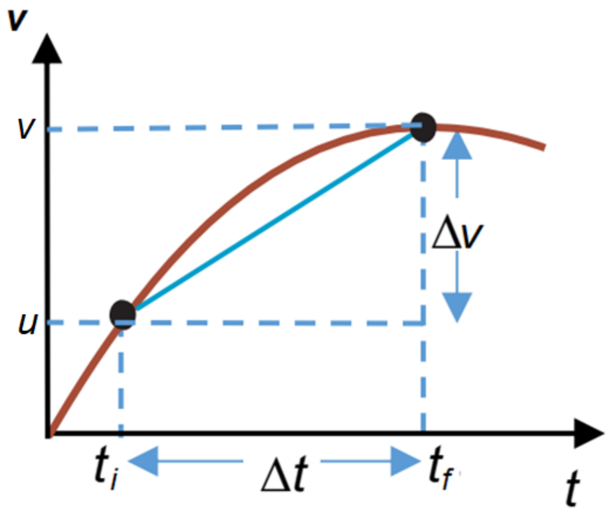
Acceleration

Symbol: $\vec{a}$
SI unit: m/s^2
Vector

Average Acceleration

$$\vec{a}_{ave} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v} - \vec{u}}{t_f - t_i}$$

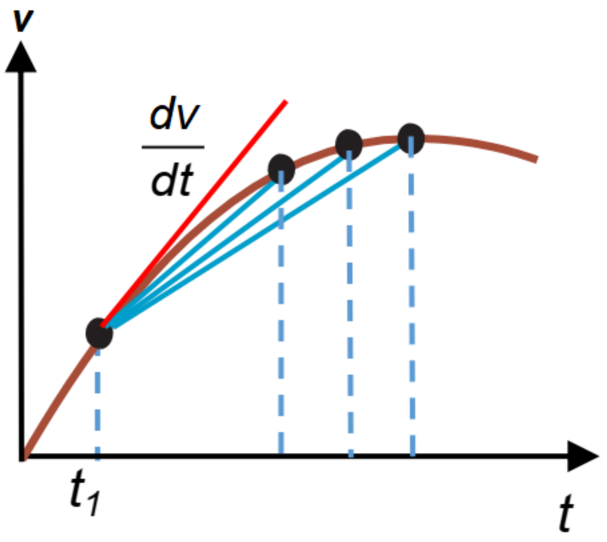
$\vec{v}$ = final velocity; $\vec{u}$ = initial velocity



NOTE: The area under the graph of an v vs t graph will give the change in velocity.
$\int a\, dt = \Delta v$

Instantaneous Acceleration

$$\vec{a} = \frac{d\vec{v}}{dt}$$



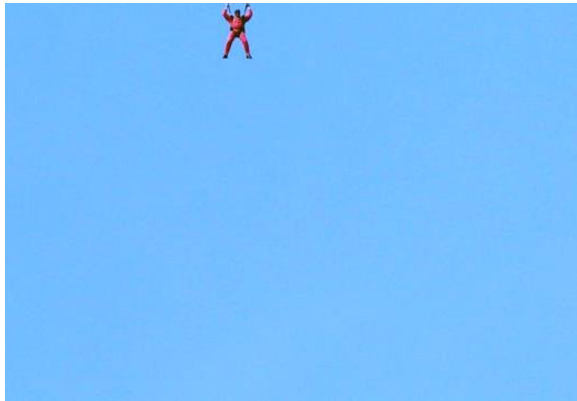
Instantaneous acceleration at a point in time t is the slope of the tangent line in a v vs t plot.

Positive Acceleration

- speeding up in the positive direction
- slowing in the negative direction

Negative Acceleration

- slowing down in the positive direction
- speeding up in the negative direction



Practice Example 1

Tonette went out for a run. From her house, she jogged towards east for 7 minutes covering 0.75 meters. At this point, she realized that she forgot to bring her water bottle. She went back home and grabbed her water bottle which took additional 7 minutes. She continued her jog eastward and ended up 1.5 km from her house. This took 20 minutes. She turned back, jogged for additional 30 minutes towards west. She covered 2.25 km for this last leg as she stopped to rest.

a) What is the total distance Tonette travelled?
b) What is Tonette's displacement?
c) Determine the average velocity of her entire run.
d) Plot Tonette's position versus time.

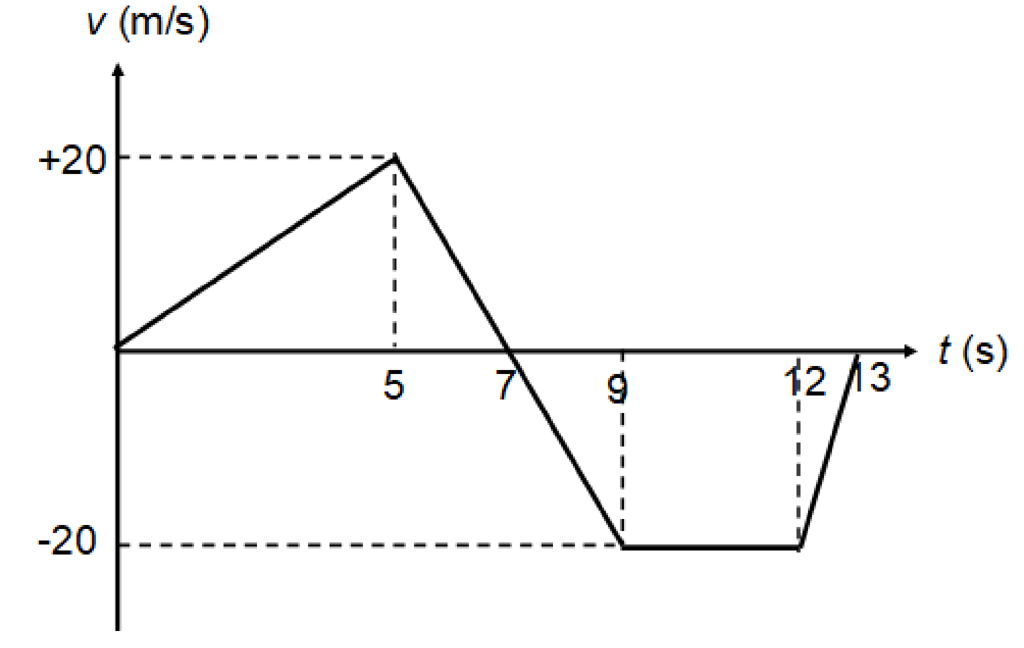
Practice Example 2

An automobile, travelling in a straight expressway, starts from rest at $x = 0, t = 0$. It travels with a speed of 20 m/s as it passes the point $x = 100m$ at $t = 5s$. The speed picks up at $300m/s$ when it passes the point $x = 500m$ at $t = 30s$.

a) Determine the average velocity between *t=5.0* s and *t=30.0 s.*
b) Find the average acceleration between *t=5.0* s and *t=30.0 s.*

Practice Example 3

The graph below shows the time evolution of the velocity of a car. Using the graph, find



a) acceleration of the car at $t = 5.0\ s$.
b) velocity at $t = 3.0\ s$.
c) displacement of the car from $t = 5.0\ s$ to $t = 12.0\ s$.
d) average speed for the first 9.0 s.
e) average velocity of the car from $t = 5s$ to $t = 12.0s$
f) Did the car change direction? If it did, at which point in time did it happen?

Equations of Motion

Uniformly Accelerated Motion

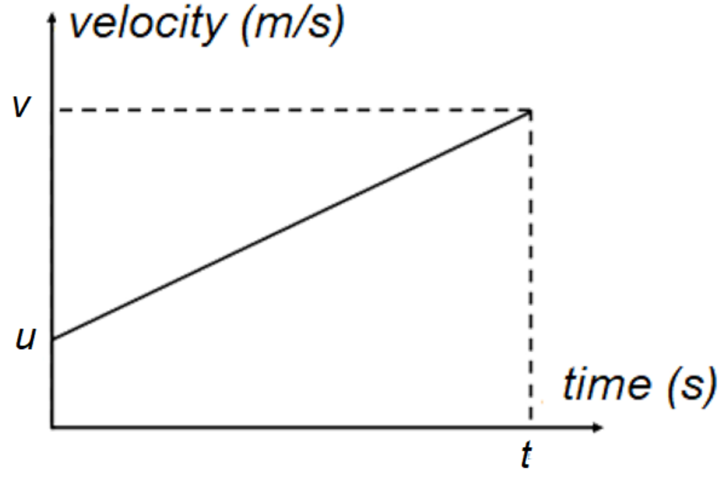
- Motion of an object whose velocity is increasing at a steady rate

$$v = u + at$$
$$s = ut + \frac{1}{2}at^2$$
$$v^2 = u^2 + 2as$$
$$v_{ave} = \frac{v+u}{2}$$

Kinematics Equations of Motion

For constant acceleration a

u = initial velocity
v = final velocity
s = displacement
a = acceleration
(must be constant)
t = time



Acceleration is the gradient of a v-t graph, so for uniformly accelerated motion, the plot is linear.

NOTE: The equations of motion are in fact vector equations. For all rectilinear vector problems, we can indicate the directions using either **positive** or **negative** **sign.**

Equations of Motion

Derivations

Since acceleration is defined as the rate of change of velocity

$$a = \frac{v-u}{t}$$

Rearranging,

$$v = u + at \qquad (1)$$

The displacement s is the area under the curve

$$s = \frac{1}{2}(v+u)t \qquad (2)$$

Subsituting equation (1) into (2)

$$s = \frac{1}{2}[(u+at)+u]t \quad \rightarrow \quad s = \frac{1}{2}[\,2u+at]t$$

$$s = ut + \frac{1}{2}at^2 \qquad (3)$$

Using equation (1)

$$t = \frac{v-u}{a} \qquad (4)$$

Subsituting equation (4) into (3)

$$s = u\left(\frac{v-u}{a}\right) + \frac{1}{2}a\left(\frac{v-u}{a}\right)^2$$

This can be simplified as

$$v^2 = u^2 + 2as \qquad (5)$$

Equations of Motion

Free Falling Body

- Object falling under the sole influence of gravity (no air resistance)
- Example of object moving in uniformly accelerated motion

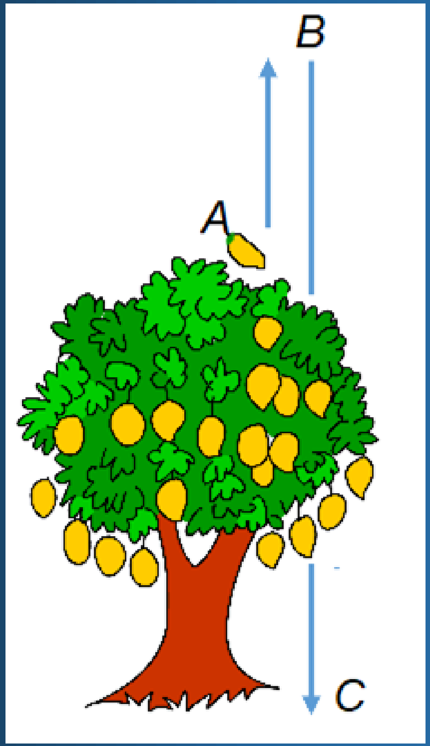
Regardless of whether the object is thrown upwards or downwards, or released from rest, the object experiences an **acceleration directed downwards** (toward the center of the Earth) with magnitude $g = 9.81 m/s^2$.

- If we take the upward motion as positive $a = -g$
- g is always $+9.81$ m/s^2

Practice Example 4

Jetliners typical take off speed ranges from 240-286 km/hr. For light aircrafts which can accelerate at 3.0 m/s^2, take off speed is around 27.8 m/s (100km/hr).

a) Can the light aircraft reach the take-off speed if the runway is 100-m long?
b) If not, what is the minimum length of the runway to reach this required speed?



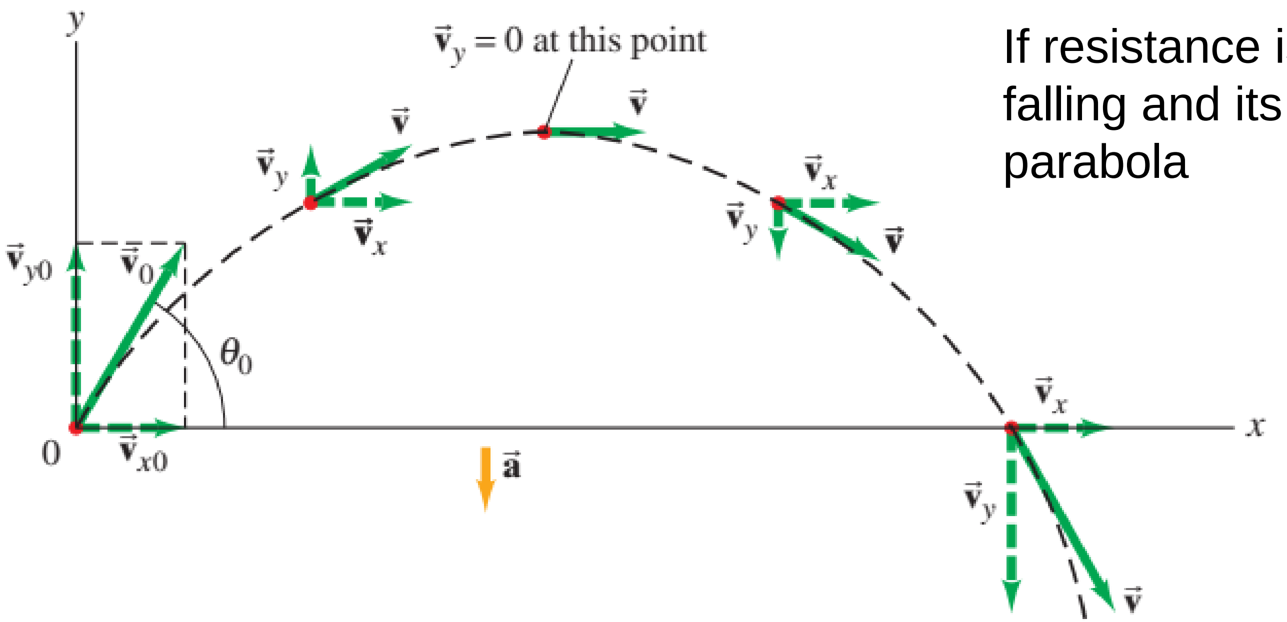
Practice Example 5

On top of a mango tree which has a height of 3.0 m, you threw a newly picked mango upwards with a speed of 12.0 m/s. Neglecting air resistance, calculate the following:

a) the time taken to reach the maximum height.
b) the maximum height measured from the ground.
c) the time taken for the mango to reach the ground.
d) speed of the mango before it hit the ground

Projectile Motion

- is a motion of an object thrown into air, subject only to acceleration due to gravity



If resistance is ignored, a projectile is free falling and its path follows a shape of parabola

Main Characteristics

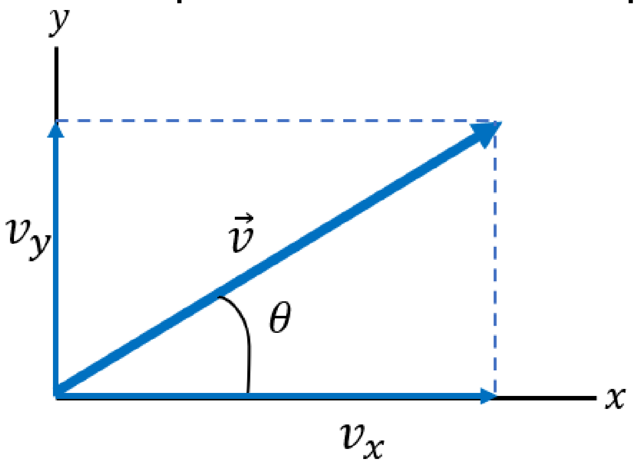
1. Constant horizontal velocity
2. Uniform vertical acceleration

Projectile Motion

Resolving Velocities

Projectile motion can be understood by analyzing the horizontal (x-axis) and vertical (y-axis) motion separately. That is, displacement, velocity and acceleration of a projectile have their horizontal and vertical components: $\vec{s} \rightarrow s_x, s_y$, $\vec{v} \rightarrow v_x, v_y$, $\vec{a} \rightarrow a_x, a_y$.

For example, a vector $\vec{v}$ which points at an angle θ from the horizontal has components:



$$v_x = v\cos\theta\,,$$
$$v_y = v\sin\theta$$

where the magnitude and angle can is given by

$$v = \sqrt{v_x^2 + v_y^2}$$
$$\tan\theta = \frac{v_y}{v_x} = \frac{\sin\theta}{\cos\theta}$$

NOTE:

- A projectile motion exhibits constant horizontal velocity which means that $a_x = 0$ and $v_x = u\cos\theta_0$ (where θ_0 is the launch angle of the projectile)
- It has uniform vertical motion implying $a_y = -g = -9.81\text{m/s}^2$

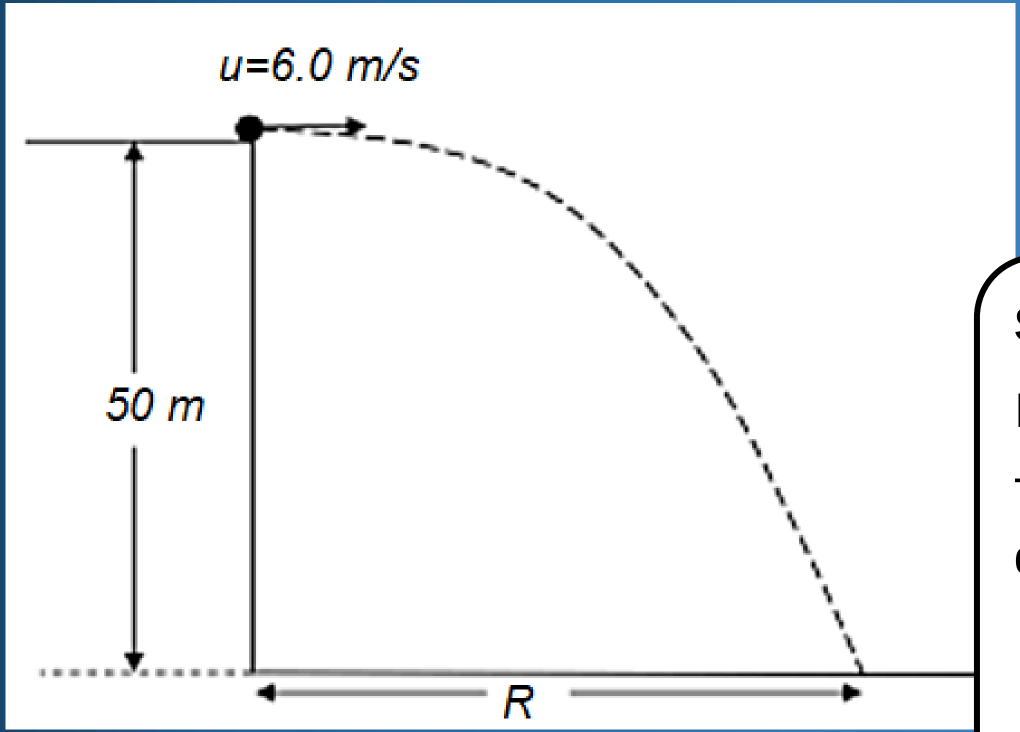
Projectile Motion

Sign Conventionality	
Displacement	**Positive**: the object's final position is **above** or at the **right** of its initial position **Negative**: below or at the left of the initial position
Velocity	**Positive**: the object is moving **upwards** or to the **right** **Negative**: motion is downwards or to the left
Acceleration	$a_x = 0$ $a_y = -g = -9.81$ m/s^2

<table>
<tr><th colspan="3">Range and Maximum Height</th></tr>
<tr><td>Range</td><td>Horizontal distance travelled by the projectile
$R = \dfrac{u^2 \sin(2\theta_0)}{g}$</td><td rowspan="2">Valid only for projectile that lands on the same level it started</td></tr>
<tr><td>Maximum Height</td><td>$h = \dfrac{u^2 \sin^2 \theta_0}{2g}$
$u =$ initial speed
$\theta_0 =$ launch angle</td></tr>
<tr><td>Note</td><td colspan="2">At the maximum height, the vertical velocity is zero.
$v_y = 0$</td></tr>
</table>

Worked Example

A stone is projected with and initial horizontal velocity u_x of 6.0 m/s at a height of 50 m above the ground as shown in the figure below. Assume negligible air resistance.



a) Determine the time it takes for the stone to reach the ground after projection

Solution:

Initial velocity components: $u_x = u\cos 0 = 6.0m/s, \qquad u_y = 0$

The final position is below the initial position so the vertical displacement must be negative.

$$s_y = u_y t - \frac{1}{2}gt^2 \rightarrow -50 = 0 - \frac{1}{2}(9.81)t^2$$

$$t = \pm 3.19s$$

We consider the positive time: $t = 3.19s$

Worked Example (continuation)

b) Determine the range R of the stone.

Solution:
We cannot use the equation above to solve R because the stone did not land at the same level it started.

$$R \rightarrow s_x = u_x t = 6.0(3.19s) = 19.2m$$

c) What is the velocity of the stone before it hits the ground?

Solution: $v_y^2 = u_y^2 + 2a_y s_y = 0 + 2(-9.81)(-50) \quad \Rightarrow \quad v_y = \pm 31.3m/s$

Since the motion is downwards, $v_y = -31.3m/s$

Alternatively, $v_y = u_y - gt = 0 - 9.81(3.19) = -31.3m/s$

d) At a certain t_1, the velocity of the stone v is 17 m/s at an angle $69°$ below horizontal. Determine the magnitude of the change in velocity between $t = 0$ and $t = t_1$

Solution: At $t = 0$, $u_x = u\cos 0 = 6.0m/s,$ $\quad u_y = 0$

At $t = t_1$, $v_x = v\cos 69 = 6.09$m/s, $\quad v_y = v\sin 69 = 15.87\, m/s$

x component: $\Delta v_x = v_x - u_x = 0.09$m/s

y component: $\Delta v_y = v_y - u_y = 15.87$m/s

magnitude: $\Delta v = \sqrt{\Delta v_x^2 + \Delta v_y^2} = 15.87$m/s

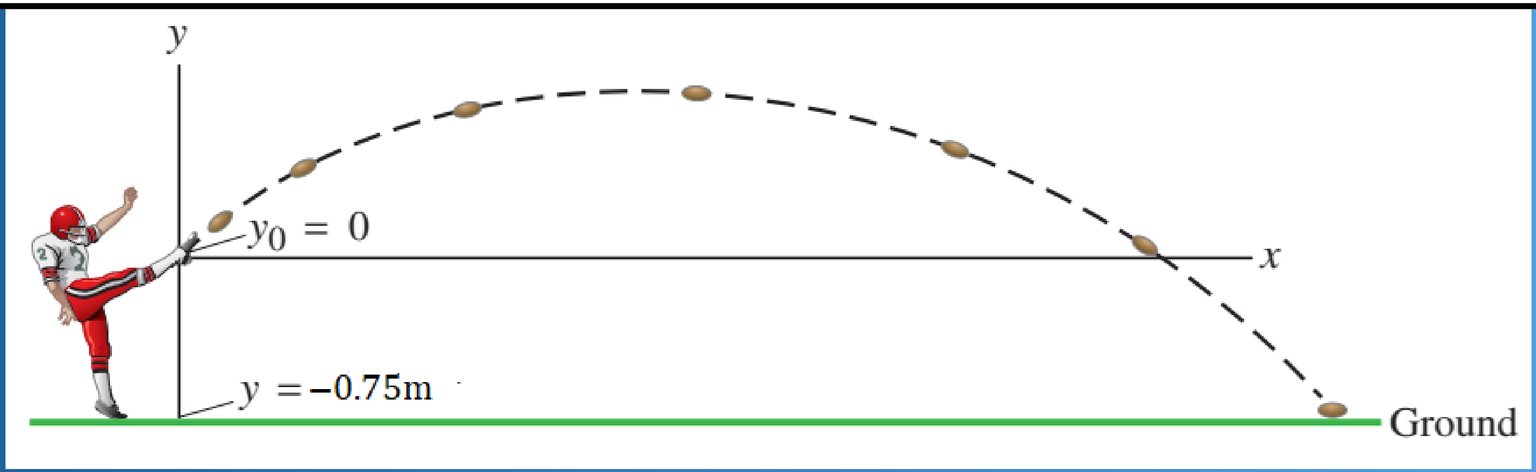
Practice Example 6

A ball is launched at angle of $30°$ from the ground having a speed of 25.0 m/s.

a) Determine the time taken for the ball to reach its maximum height
b) What is the maximum height of the ball?
c) Solve the time taken by the ball to reach the ground.
d) What is the speed of the ball before it hits the ground?

Practice Example 7

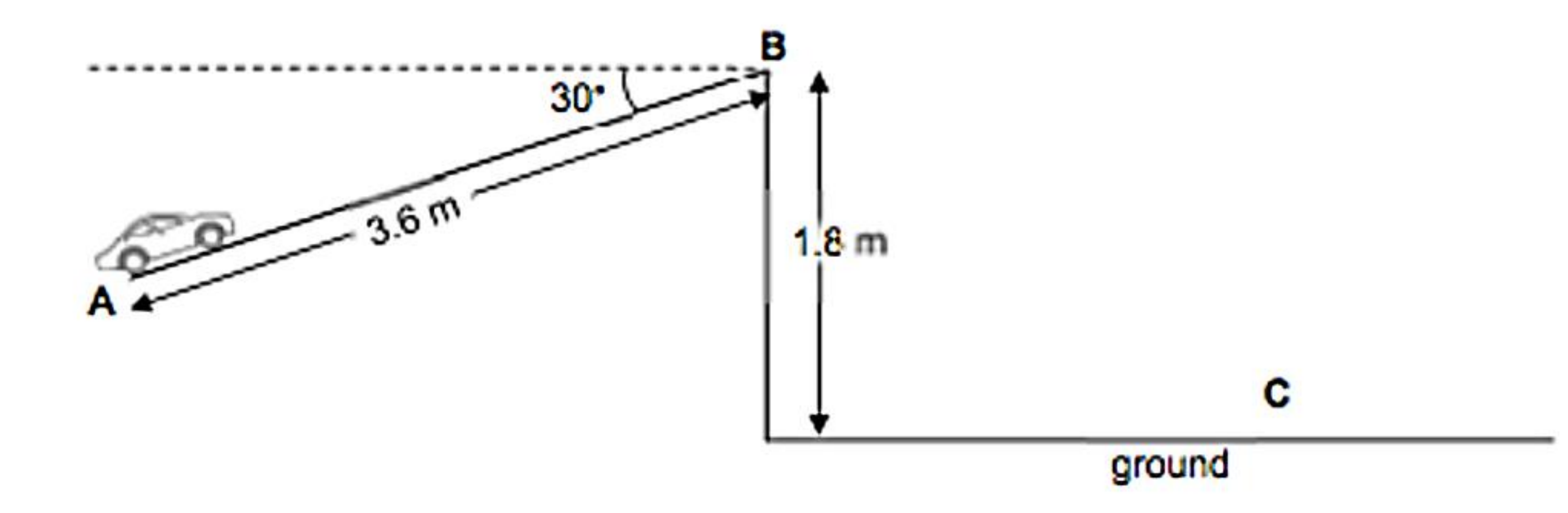
A football player kicks a ball at a height of 0.75 m from the ground. It leaves his foot at angle of $32°$ with a speed of 22.0 m/s. Let the initial position of the ball be the origin of your coordinate system.



a) What is the maximum height of the ball measured from the ground?
b) How far did the ball travel before it hit the ground?

Practice Example 8

A motorized toy car moves from point **A** from rest with uniform acceleration of 0.20 m/s^2 along a slope at an angle of 30° to the horizontal, as shown in the figure below. It travels 3.6 m along the slope before free falling to the ground at a vertical distance of 1.8 m below point **B.** The toy car is just about to hit the ground at point **C.** Neglect air resistance**.**



a) Show that the speed of the car at **B** is 1.2 m/s.
b) Calculate the time taken for the car to travel from point **A** to **B.**
c) Calculate the time taken for the car to move from point **B** to **C.**
d) Calculate the horizontal displacement of the toy car from point **B** to **C**

Motion in a Uniform Gravitational Field with Air Resistance

- When an object moves through liquid, it experiences **viscous drag.**
- When an object moves through air, it experiences **air resistance.**
- When there is air resistance, falling objects do not have the same acceleration of a free-fall

Drag Force

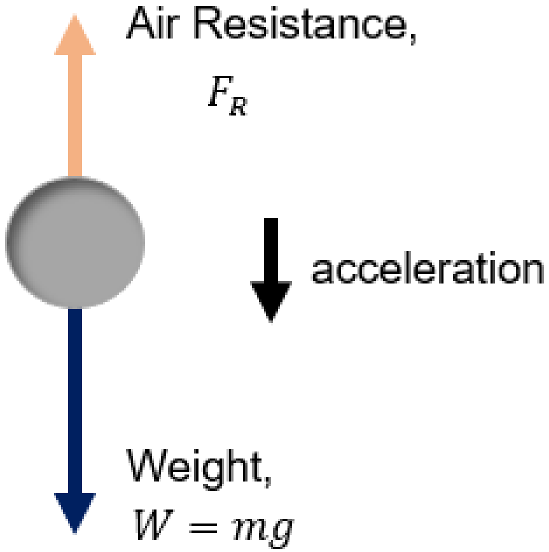
At low velocities with no turbulence, the magnitude of drag force is

$$|F_D| = k\,\vec{v}$$

At higher velocities with turbulence,

$$|F_D| = k\,|\vec{v}\,|^2$$

k = constant, $|\vec{v}|$ = magnitude of velocity



Free- Fall

Consider a falling ball. Using
which implies that the acceleNewton's 2nd law,

$$F_{net} = ma$$

$$mg - F_R = ma$$

$a = g - \frac{F_R}{m}$ration of the ball is less than the acceleration of a free-falling body.

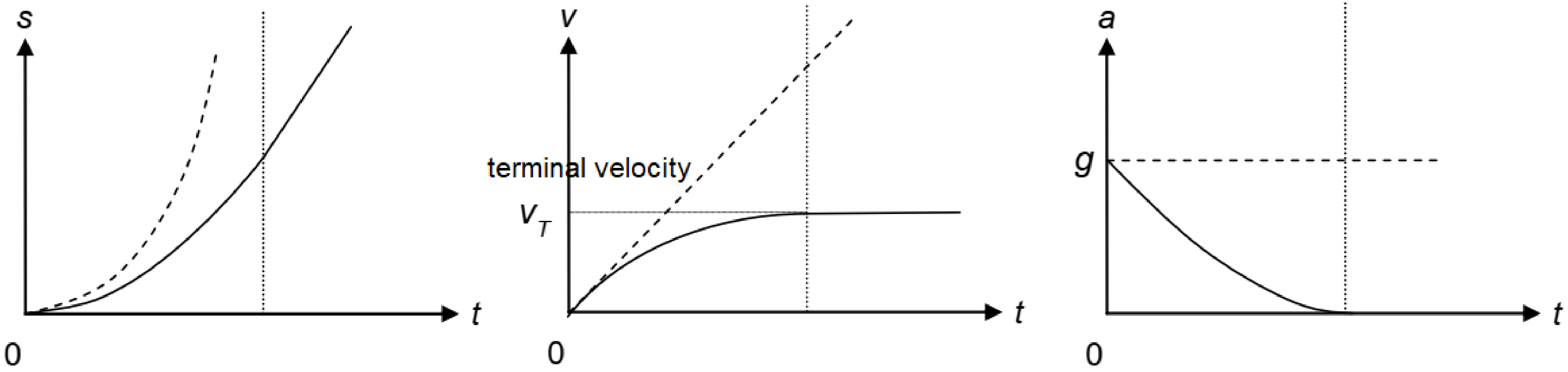
Motion in a Uniform Gravitational Field with Air Resistance

- As the object falls, its velocity increases resulting to stronger effect of air resistance.
- This opposes the downward motion causing a decrease in acceleration.

$$a = g - \frac{F_R}{m}$$

- There will be a point where the weight is equal to the drag force $F_D = mg$.
- The object experiences zero acceleration and now falls with a constant velocity called as **terminal velocity**.
- The drag force does not increase any further since velocity is constant.

Graph of displacement, velocity, and acceleration with respect to time of a falling body with and without air resistance

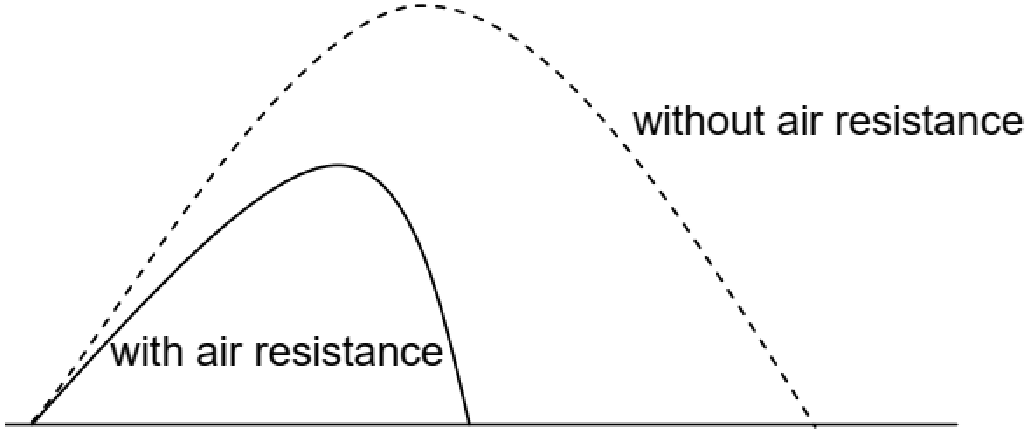


*dotted line indicates the time where velocity and air resistance do not increase any further

Motion in a Uniform Gravitational Field with Air Resistance

Projectile

In the presence of air resistance, the body experiences drag force in the direction opposite to its motion.



- As it travels up, both drag force and weight act downwards which means the net downward force is larger.
- By Newton's 2nd law, the downward acceleration is also larger than $g = 9.8\text{m/s}^2$.
- This causes the projectile to reach a **smaller altitude** at **sooner time**.
- Also, since there is air resistance acting horizontally, the horizontal velocity is reduced resulting to **smaller range**.

Note: Equations of motions discussed above are not applicable if air resistance is present

Suggested Solutions to Practice Examples

Practice Example 1

East (+ direction), West (– direction)

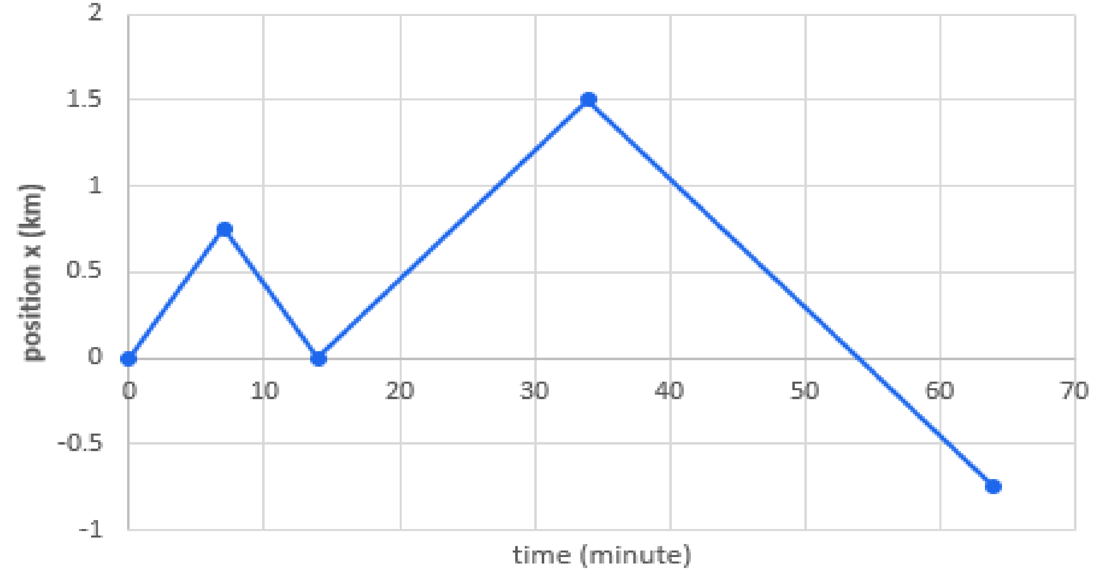
i	Time t_i (min)	Position x_i (km)	Displacement (km)
0	0	0	$\Delta x_1 = 0$
1	7	0.75	$\Delta x_1 = x_1 - x_0 = 0.75$
2	14	0	$\Delta x_2 = x_2 - x_1 = -0.75$
3	34	1.5	$\Delta x_3 = x_3 - x_2 = 1.5$
4	64	-0.75	$\Delta x_4 = x_4 - x_3 = -2.25$

a) $\mathrm{d} = \sum_i |x_i| = (0 + 0.75 + 0.75 + 1.50 + 2.25)\mathrm{km} = 5.25\mathrm{km}$

b) $\vec{s} = \vec{x_f} - \vec{x_i} = \vec{x_4} - \vec{x_1} = -0.25\mathrm{km} - 0\mathrm{km} = -0.25\mathrm{km}$ (negative means the direction of displacement is westward). Equivalently $\vec{s} = 0.25\mathrm{km}$, westward

c) Average velocity

$$\vec{v}_{ave} = \frac{\Delta\vec{x}}{\Delta t} = \frac{\vec{s}}{\Delta t} = \frac{\vec{s}}{t_4 - t_0} = \frac{-0.25km}{64min} = -3.91 \times 10^{-3}\frac{km}{min} = -65.1 \times 10^{-3} m/s$$



Practice Example 2

a) $\vec{v}_{ave} = \frac{\Delta\vec{x}}{\Delta \text{t}} = \frac{500m-100m}{30s-5s} = 16m/s$ in the positive direction

b) $\vec{a}_{ave} = \frac{\Delta\vec{v}}{\Delta \text{t}} = \frac{(300-20)m/s}{30s-5s} = 11.2m/s$ in the positive direction

Practice Example 3

a. $a = \frac{v}{t} = \frac{20m/s}{5s} = 4.0m/s^2$

b. $\frac{v}{3} = \frac{20}{5} \rightarrow v = 12m/s$

c. Area under the curve in a v vs t plot gives the displacement.

$$d = A_{5-7} + A_{7-9} + A_{9-12}$$

$$\vec{s} = \frac{1}{2}(20)(7-5) + \frac{1}{2}(-20)(9-7) + (-20)(12-9) = -60m$$

d. $v_{ave} = \frac{d}{\Delta t} = \frac{distance\ travelled}{time\ taken}$

$$v_{ave} = \frac{\frac{1}{2}(20)(5) + \frac{1}{2}(20)(2) + \frac{1}{2}(20)(2)}{9s} = 6.94m/s$$

e. Yes, at $t = 7.0s$

Practice Example 4

a. Given: $a = 3.0\ m/s^2$,$u = 0$, $v_{takeoff} = 27.8m/s$

For $s = 100$m, we use equation (5) to solve for v. If $v \geq 27.8m/s$, then the runway length is enough to reach the required take-off speed.

$$v^2 = v_0^2 + 2as = 0 + 2(3)(100) = 1200$$
$$v = 24.49\ m/s$$

Ans: No.

b. $v^2 = u^2 + 2as \rightarrow s = \frac{v^2 - u^2}{2a}$

$$s = \frac{27.8^2 - 0}{2(3)} = 128.8m$$

Practice Example 5

We take the upward motion as the positive (+) direction.

a. From point A to B, acceleration is decreasing in the positive direction hence a must be negative. $a = -g = -9.8\text{m/s}^2$. At the maximum height, the velocity is zero, $v = 0$.

At point A (our initial point), $u = 12m/s$

$$v = u - gt \rightarrow t = \frac{u - v}{g} = \frac{12 - 0}{9.8} = 1.22s$$

b. Finding for s

$$v^2 = u^2 + 2as \rightarrow s = \frac{v^2 - u^2}{-2g} = \frac{0^2 - 12^2}{-2(9.8)} = 7.35m$$

Note that this is the displacement as measured from the top of the tree. To determine the maximum height measured from the ground, we add the height of the tree.

$$y_{max} = 7.35 + 3 = 10.35m$$

c. The displacement is negative since the final position is below the

$$s = ut + \frac{1}{2}at^2 \rightarrow -3 = 12t - \frac{1}{2}(9.8)t^2$$

Using quadratic equation $t = 2.68s \; or - 0.23s$

Ans: t=2.68 s (we reject the negative t)

d. $v = u + at = 12 - 9.8(2.68) = -14.26m/s$

Practice Example 6

a. Initial velocity components: $u_x = u\cos\theta_0 = 21.65m/s,$ $\quad u_x = 12.5m/s$

At max height, $v_y = 0,$ $\quad v_x = u_x = 21.65m/s.$

$$v_y = u_y - gt \to t = \frac{u_y - v_{0y}}{-g} = \frac{0 - 12.5}{-9.8} = 1.28s$$

b. $h = \frac{u^2\sin^2\theta_0}{2g} = \frac{25^2\sin^2 30}{2(9.8)} = 7.97m$

or use the result form part (a): $s_y = u_y t - \frac{1}{2}gt^2 = 12.5(1.28) - 4.9(1.28)^2 = 7.97m$

c. $y = 0$ (ground): $s_y = u_y t - \frac{1}{2}gt^2 \to 0 = 12.5t - 4.9t^2$

Using quadratic formula: $t = 0$ or 2.55

d. $v_x = u_x = 21.65$m/s

As for v_y, we use

$$v_y = u_y - gt = 12.5 - (9.8)(2.55) = -12.49m/s$$

$$v = \sqrt{21.65^2 + 12.49^2} = 24.99m/s$$

Practice Example 7

a. Initial velocity components: $u_x = u\cos\theta = 18.65m/s,\qquad u_y = 11.65m/s$

At max height $v_y = 0$,

$$t = \frac{v_y - u_y}{-g} = \frac{0 - 11.65}{-9.8} = 1.19s$$

$$s_y = u_y t - \frac{1}{2}gt^2 = 0 + 11.65(1.19) - 4.9(1.19)^2 = 6.92m$$

b. $s_y = -0.75m$ (ground level which is below the initial position)

$$s_y = u_y t - \frac{1}{2}gt^2 \rightarrow -0.75 = 11.65t - 4.9t^2$$

$$0 = 0.75 + 11.65t - 4.9t^2$$

Using quadratic formula, we get the time taken for the ball to hit the ground.

$$t = -0.0627s \text{ or } t = 2.44s$$

Only considering the positive time t,

$$x = x_0 + u_x t = 0 + 18.65(2.44) = 45.51m$$

Practice Example 8

a. From point **A** to **B,** the motion is a uniformly accelerated motion. Let's denote all variables with a prime (') to differentiate it with the projectile motion which happens between point **B** and **C.**

$$v'^2 = u'^2 + 2as = 0 + 2(0.2)(3.6) = 1.44$$

$$v' = 1.2\text{m/s}$$

Note that v' will be the u for the projectile motion from point **B** to **C.**

b. $s' = u't + \frac{1}{2}a't^2 \rightarrow 3.60 = 0 + \frac{1}{2}(0.2)t^2$

$$t = \pm 0.60s$$

Rejecting the negative result. $t = 0.60s$

c. Initial velocity at point B: $u_x = u\cos\theta = 1.2\cos 30 = 1.04\,m/s$

$$u_y = u\sin\theta = 1.2\sin 30 = 0.6m/s$$

$$s_y = u_y t - \frac{1}{2}gt^2 \rightarrow -1.8 = 0.6t - \frac{1}{2}(9.81)t^2 \rightarrow 0 = 1.8 + 0.6t - 4.905t^2$$

Using quadratic formula $t = -0.55$ or $0.67s$

d. $s_x = v_x t = u_x t = 1.04(0.67) = 0.70m$