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February Practice Questions 2022 Full Solutions (A-Math)

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Question Source

All questions are sourced and selected based on the known abilities of students sitting for the 'O' Level A-Math Examination. All questions compiled here are from **2009 - 2021 School Mid-Year / Prelim Papers**. Questions are categorised into the various topics and range in varying difficulties. If questions are sourced from respective sources, credit will be given when appropriate.

How to read:

[S4 ABCSS P1/2011 PRELIM Qn 1]

Secondary 4, ABC Secondary School, Paper 1, 2011, Prelim, Question 1

Syllabus (4049)

Algebra	**Geometry and Trigonometry**	**Calculus**
Quadratic Equations & Inequalities	Trigonometry	Differentiation
Surds	Coordinate Geometry	Integration
Polynomials	Further Coordinate Geometry	Kinematics
Simultaneous Equations	Linear Law	
Partial Fractions	Proofs of Plane Geometry	
Binomial Theorem		
Exponential & Logarithms		

Contents

1 Quadratic Equations & Inequalities

1.1 Full Solutions

1. (a) Since the expression is never negative, $b^2 - 4ac < 0$

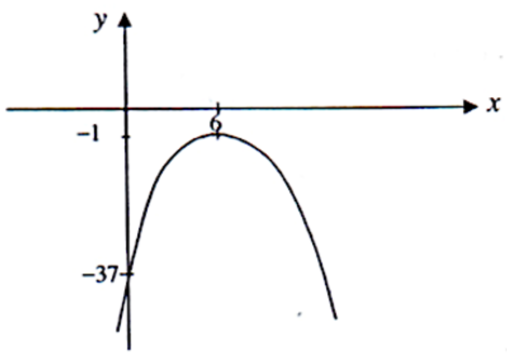
$$(-2p)^2 - 4(1)\left(2p^2 - \frac{1}{4}(5p+6)\right) < 0$$
$$4p^2 - 4\left(2p^2 - \frac{5}{4}p - \frac{6}{4}\right) < 0$$
$$4p^2 - 8p^2 + 5p + 6 < 0$$
$$4p^2 - 5p - 6 > 0$$
$$(4p+3)(p-2) > 0$$

$$\therefore \boldsymbol{p < -\frac{3}{4} \quad \text{or} \quad p > 2}$$

(b) (i) By completing the square,

$$\begin{aligned} -x^2 + 12x - 37 &= -\left(x^2 - 12x + 37\right) \\ &= -\left[(x-6)^2 - 36 + 37\right] \\ &= \boldsymbol{-(x-6)^2 - 1} \end{aligned}$$

(ii) Curve of $y = -x^2 + 12x - 37$



(iii) Range of y:

$$\boldsymbol{y \leq -1}$$

2.

$$(x-a)(b-x) = m$$
$$xb - x^2 - ab + ax - m = 0$$
$$-x^2 + (a+b)x - ab - m = 0$$
$$x^2 - (a+b)x + (ab+m) = 0$$

Since the roots are equal, $b^2 - 4ac = 0$

$$(a+b)^2 - 4(1)(ab+m) = 0$$
$$a^2 + 2ab + b^2 - 4ab - 4m = 0$$
$$a^2 - 2ab + b^2 - 4m = 0$$
$$(a-b)^2 - 4m = 0$$

$$\boldsymbol{m = \left(\frac{a-b}{2}\right)^2} \textbf{ (shown)}$$

3. (a)

$$px^2 + 4x + p > 3$$
$$px^2 + 4x + (p-3) > 0$$

Since the quadratic equation is strictly positive, $b^2 - 4ac < 0$

$$(4)^2 - 4(p)(p-3) < 0$$
$$16 - 4p^2 + 12p < 0$$
$$4p^2 - 12p - 16 > 0$$
$$p^2 - 3p - 4 > 0$$
$$(p-4)(p+1) > 0$$
$$\therefore p > 4 \quad \text{or} \quad p < -1$$

Note that a condition for the expression to always be positive is that the coefficient of x^2 must always be positive

$$\therefore \boldsymbol{p > 4}$$

(b)

Equation 1: $x = k - 5y$

Equation 2: $5x^2 + 5xy + 4 = 0$

Substitute Equation (1) into Equation (2),

$$5(k-5y)^2 + 5(k-5y)y + 4 = 0$$
$$5k^2 - 50ky + 125y^2 + 5ky - 25y^2 + 4 = 0$$
$$100y^2 - 45ky + \left(5k^2 + 4\right) = 0$$

Since the line does not intersect the curve, $b^2 - 4ac < 0$

$$(45k)^2 - 4(100)\left(5k^2 + 4\right) < 0$$
$$2025k^2 - 2000k^2 - 1600 < 0$$
$$k^2 - 64 < 0$$
$$\therefore \boldsymbol{-8 < k < 8}$$

4.

$$y = x^2 \text{ ......(1)}$$
$$y = px - q^2 \text{ ......(2)}$$

Let Equation (1) = Equation (2),

$$x^2 = px - q^2$$
$$x^2 - px + q^2 = 0$$

Since the curve lies above the line, there is no intersection, $b^2 - 4ac < 0$

$$(-p)^2 - 4(1)\left(q^2\right) < 0$$
$$p^2 - 4q^2 < 0$$

From the given range,

$$-2 < p < 2$$

$$(p-2)(p+2) < 0$$
$$p^2 - 4 < 0$$

By comparison,

$$4q^2 = 4$$
$$\boldsymbol{q = \pm 1}$$

2 Surds

2.1 Full Solutions

1. (a) We first solve for $\left(1-\sqrt{a}\right)^5$,

$$\begin{aligned}
\left(1-\sqrt{a}\right)^2 &= 1-2\sqrt{a}+a \\
\left(1-\sqrt{a}\right)^4 &= \left(1-2\sqrt{a}+a\right)^2 \\
&= 1-2\sqrt{a}+a-2\sqrt{a}+4a-2a\sqrt{a}+a-2a\sqrt{a}+a^2 \\
&= 1-4\sqrt{a}-4a\sqrt{a}+6a+a^2 \\
\therefore \left(1-\sqrt{a}\right)^5 &= \left(1-4\sqrt{a}-4a\sqrt{a}+6a+a^2\right)\left(1-\sqrt{a}\right) \\
&= 1-\sqrt{a}-4\sqrt{a}+4a-4a\sqrt{a}+4a^2+6a-6a\sqrt{a}+a^2-a^2\sqrt{a} \\
&= 1-5\sqrt{a}-10a\sqrt{a}-a^2\sqrt{a}+10a+5a^2
\end{aligned}$$

Next, for $\left(1+\sqrt{a}\right)^5$, by inspection,

$$\left(1+\sqrt{a}\right)^5 = 1+5\sqrt{a}+10a\sqrt{a}+a^2\sqrt{a}+10a+5a^2$$

$$\begin{aligned}
&\therefore \left(1-\sqrt{a}\right)^5-\left(1+\sqrt{a}\right)^5 \\
&= \left[1-5\sqrt{a}-10a\sqrt{a}-a^2\sqrt{a}+10a+5a^2\right]-\left[1+5\sqrt{a}+10a\sqrt{a}+a^2\sqrt{a}+10a+5a^2\right] \\
&= 1-5\sqrt{a}-10a\sqrt{a}-a^2\sqrt{a}+10a+5a^2-1-5\sqrt{a}-10a\sqrt{a}-a^2\sqrt{a}-10a-5a^2 \\
&= \mathbf{-10\sqrt{a}-20a\sqrt{a}-2a^2\sqrt{a}\ \text{(shown)}}
\end{aligned}$$

(b) By comparing part (a) and (b),

$$a = 3$$

$$\begin{aligned}
\therefore \left(1-\sqrt{3}\right)^5-\left(1+\sqrt{3}\right)^5 &= -10\sqrt{3}-20(3)\sqrt{3}-2(3)^2\sqrt{3} \\
&= \mathbf{-88\sqrt{3}}
\end{aligned}$$

Alternative method for part (a)

The intial part of the question can also be solved using the Binomial Theorem

$$\begin{aligned}
&\left(1-\sqrt{a}\right)^5 \\
&= 1+\binom{5}{1}\left(-\sqrt{a}\right)+\binom{5}{2}\left(-\sqrt{a}\right)^2+\binom{5}{3}\left(-\sqrt{a}\right)^3+\binom{5}{4}\left(-\sqrt{a}\right)^4+\left(-\sqrt{a}\right)^5 \\
&= 1-5\sqrt{a}+10a-10a^{1\frac{1}{2}}+5a^2-a^{2\frac{1}{2}} \\
&= 1-5a-10a\sqrt{a}-a^2\sqrt{a}+10a+5a^2
\end{aligned}$$

The remaining part of the question is the same

2. Using the volume formula for geometrically similar solids,

$$\frac{V_{small}}{V_{large}} = \left(\frac{l_{small}}{l_{large}}\right)^3$$

$$\frac{1}{2\sqrt{2}} = \left(\frac{\frac{3+2\sqrt{2}}{\left(1-\sqrt{2}\right)^2}}{l_{large}}\right)^3$$

$$\begin{aligned}
\therefore (l_{large})^3 &= 2\sqrt{2}\left(\frac{3+2\sqrt{2}}{\left(1-\sqrt{2}\right)^2}\right)^3 \\
&= 2\sqrt{2}\left(\frac{3+2\sqrt{2}}{1-2\sqrt{2}+2}\right)^3 \\
&= 2\sqrt{2}\left(\frac{3+2\sqrt{2}}{3-2\sqrt{2}}\right)^3 \\
&= 2\sqrt{2}\left(\frac{3+2\sqrt{2}}{3-2\sqrt{2}} \times \frac{3+2\sqrt{2}}{3+2\sqrt{2}}\right)^3 \\
&= 2\sqrt{2}\left(9+12\sqrt{2}+8\right)^3 \\
&= 2\sqrt{2}\left(17+12\sqrt{2}\right)^3 \\
&= \sqrt{8}\left(17+12\sqrt{2}\right)^3
\end{aligned}$$

$$\begin{aligned}
\therefore l_{large} &= \sqrt[3]{\sqrt{8}\left(17+12\sqrt{2}\right)^3} \\
&= \sqrt{2}\left(17+12\sqrt{2}\right) \\
&= \mathbf{17\sqrt{2}+24\ cm}
\end{aligned}$$

3. (a)

$$\begin{aligned}\left(\frac{4}{2+\sqrt{5}}-3-2\sqrt{5}\right)^2 &= \left(\frac{4}{2+\sqrt{5}}\times\frac{2-\sqrt{5}}{2-\sqrt{5}}-3-2\sqrt{5}\right)^2\\ &= \left(\frac{8-4\sqrt{5}}{-1}-3-2\sqrt{5}\right)^2\\ &= \left(4\sqrt{5}-8-3-2\sqrt{5}\right)^2\\ &= \left(2\sqrt{5}-11\right)^2\\ &= 20-44\sqrt{5}+121\\ &= \mathbf{141-44\sqrt{5}}\end{aligned}$$

(b)

$$\begin{aligned}ab-4b+a-4 &= ab+a-4b-4\\ &= a(b+1)-4(b+1)\\ &= (a-4)(b+1)\end{aligned}$$

Since $6^x = 2^x \times 3^x$, let $a = 2^x$ and $b = 3^x$

$$\begin{aligned}6^x - 4\left(3^x\right) + 2^x - 4 &= 0\\ (2^x-4)(3^x+1) &= 0\end{aligned}$$

$$2^x = 4 \qquad \text{or} \qquad 3^x = -1 \text{ (rej)}$$

$$\therefore \boldsymbol{x = 2}$$

4. Using the volume formula for a prism,

$$\begin{aligned}\text{Volume of prism} &= (\text{Base Area})\,(\text{Height})\\ 11+6\sqrt{3} &= \left(2+\sqrt{3}\right)^2(\text{Height})\\ \text{Height} &= \frac{\left(11+6\sqrt{3}\right)}{\left(2+\sqrt{3}\right)^2}\\ &= \frac{11+6\sqrt{3}}{4+4\sqrt{3}+3}\\ &= \frac{11+6\sqrt{3}}{7+4\sqrt{3}}\\ &= \frac{11+6\sqrt{3}}{7+4\sqrt{3}}\times\frac{7-4\sqrt{3}}{7-4\sqrt{3}}\\ &= \frac{77-44\sqrt{3}+42\sqrt{3}-72}{1}\\ &= 5-2\sqrt{3}\end{aligned}$$

$$\therefore \text{Height} = \left(\mathbf{5-2\sqrt{3}}\right)\ \mathbf{m}$$

3 Polynomials

3.1 Full Solutions

1. (a) Since $x^2 + x - 2$ is a factor,

$$x^2 + x - 2 = (x+2)(x-1)$$

$$\therefore f(x) = (x+2)(x-1)Q_1(x)$$

Let $f(-2) = 0$

$$\begin{aligned} 3(-2)^3 + a(-2)^2 - b(-2) - 10 &= 0 \\ 4a + 2b &= 24 \\ 2a + b &= 17 \text{ ......(1)} \end{aligned}$$

Let $f(1) = 0$

$$\begin{aligned} 3(1)^3 + a(1)^2 - b(1) - 10 &= 0 \\ a - b &= 7 \\ a &= 7 + b \text{ ......(2)} \end{aligned}$$

Substitute Equation (2) into Equation (1),

$$\begin{aligned} 2(7+b) + b &= 17 \\ 3b &= 3 \\ \boldsymbol{b} &\boldsymbol{= 1} \end{aligned}$$

Substitute $b = 1$ into Equation (2),

$$\begin{aligned} a &= 7 + 1 \\ \boldsymbol{a} &\boldsymbol{= 8} \end{aligned}$$

(b) By observation,

$$\begin{aligned} f(x) &= 3x^2 + 8x^2 - x - 10 \\ &= (x^2 + x - 2)(3x + 5) \\ &= \boldsymbol{(x+2)(x-1)(3x+5)} \end{aligned}$$

(c)

$$f(x) = (2x-1)Q_2(x) + R$$

Let $x = \frac{1}{2}$,

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \left(\frac{1}{2} + 2\right)\left(\frac{1}{2} - 1\right)\left(3\left(\frac{1}{2}\right) + 5\right) \\ &= \boldsymbol{-8\frac{1}{8}} \end{aligned}$$

2. (a) (i) Since the coefficient of x^3 is 2 and the roots of the equation $f(x) = 0$ are -1, 3 and k

$$f(x) = 2(x+1)(x-3)(x-k)$$

Since $f(x)$ has a remainder of 20 when divided by $(x-4)$,

$$2(x+1)(x-3)(x-k) = (x-4)Q_1(x) + 20$$

Let $x = 4$,

$$\begin{aligned} 2\left((4)+1\right)\left((4)-3\right)\left((4)-k\right) &= 20 \\ 4-k &= 2 \\ k &= 2 \ \textbf{(shown)} \end{aligned}$$

(ii)

$$f(x) = 2(x+1)(x-3)(x-2)$$

To find the remainder when divided by $(2x-1)$,

$$2(x+1)(x-3)(x-2) = (2x-1)Q_2(x) + R$$

Let $x = \frac{1}{2}$,

$$\begin{aligned} \therefore R &= 2\left(\frac{1}{2}+1\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-2\right) \\ &= \mathbf{11\frac{1}{4}} \end{aligned}$$

(b) Given that $x^{10} - px^3 + q$ is divided by $x^2 - 1$,

$$\begin{aligned} x^{10} - px^3 + q &= \left(x^2-1\right)Q_3x + (4x+3) \\ x^{10} - px^3 + q &= (x-1)(x+1)Q_3x + (4x+3) \end{aligned}$$

Let $x = 1$,

$$\begin{aligned} (1)^{10} - p(1)^3 + q &= 4(1) + 3 \\ q - p &= 6 \ ......(1) \end{aligned}$$

Let $x = -1$,

$$\begin{aligned} (-1)^{10} - p(-1)^3 + q &= 4(-1) + 3 \\ p + q &= -2 \ ......(2) \end{aligned}$$

Take Equation (2) - Equation (1),

$$\begin{aligned} 2p &= -8 \\ \boldsymbol{p} &\boldsymbol{= -4} \end{aligned}$$

Substitute $p = -4$,

$$\begin{aligned} q - (-4) &= 6 \\ \boldsymbol{q} &\boldsymbol{= 2} \end{aligned}$$

3. (a) Since the function is divisible by $(x-2)$

$$\therefore f(x) = (x-2)Q_1(x)$$

Let $f(2) = 0$

$$\begin{aligned}(2)^3 + a(2)^2 + b(2) + 4 &= 0\\ 4a + 2b &= -12\\ 2a + b &= -6 \text{ ......(1)}\end{aligned}$$

Since the function leaves a remainder of -3 when divided by $(x+1)$

$$\therefore f(x) = (x+1)Q_2(x) - 3$$

Let $f(-1) = -3$,

$$\begin{aligned}(-1)^3 + a(-1)^2 + b(-1) + 4 &= -3\\ a - b &= -6 \text{ ......(2)}\end{aligned}$$

Take Equation (1) + Equation (2),

$$\begin{aligned}3a &= -12\\ \boldsymbol{a} &\boldsymbol{= -4}\end{aligned}$$

Substitute $a = -4$ into Equation (2),

$$\begin{aligned}-4 - b &= -6\\ \boldsymbol{b} &\boldsymbol{= 2}\end{aligned}$$

(b)

$$f(x) = x^3 - 4x^2 + 2x + 4 = (x-2)(x^2 + px - 2)$$

Comparing coefficients of x,

$$\begin{aligned}2 &= -2 - 2p\\ p &= -2\end{aligned}$$

$$\begin{aligned}\therefore f(x) &= (x-2)(x^2 - 2x - 2)\\ &= (x-2)\left[(x-1)^2 - 3\right]\\ &= \boldsymbol{(x-2)\left(x - 1 + \sqrt{3}\right)\left(x - 1 - \sqrt{3}\right)}\end{aligned}$$

4. (a) Since the function is divisible by $(x+2)$

$$\therefore f(x) = (x+2)Q_1(x)$$

Let $f(-2) = 0$

$$2(-2)^3 + a(-2)^2 + b(-2) + 8 = 0$$
$$4a - 2b = 8$$
$$2a - b = 4 \text{ ......(1)}$$

Since the function leaves a remainder of 10 when divided by $(2x-1)$

$$\therefore f(x) = (2x-1)Q_2(x) + 10$$

Let $f\left(\frac{1}{2}\right) = 10$

$$2\left(\frac{1}{2}\right)^3 + a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + 8 = 10$$
$$\frac{1}{4}a + \frac{1}{2}b = 1\frac{3}{4}$$
$$a = 7 - 2b \text{ ......(2)}$$

Substitute Equation (2) into Equation (1),

$$2(7-2b) - b = 4$$
$$\boldsymbol{b = 2}$$

Substitute $b = 2$ into Equation (2),

$$a = 7 - 2(2)$$
$$\boldsymbol{a = 3}$$

(b)

$$f(x) = 2x^3 + 3x^2 + 2x + 8 = (x+2)(2x^2 + cx + 4)$$

Comparing coefficients,

$$3x^2 = cx^2 + 4x^2$$
$$c = -1$$

$$\therefore f(x) = (x+2)(2x^2 - x + 4)$$

For $2x^2 - x + 4$,

$$b^2 - 4ac = (-1)^2 - 4(2)(4)$$
$$= -31 < 0$$

Since the discriminant value of $2x^2 - x + 4$ is less than 0, the equation has no real roots.

So $\boldsymbol{P(x) = 0}$ has only one real root. $\boldsymbol{x = -2}$

4 Partial Fractions

4.1 Full Solutions

1. (a)

$$\begin{aligned}\frac{13x-6}{2x^2+3x-9} &= \frac{13x-6}{(2x-3)(x+3)} \\ &= \frac{A}{2x-3}+\frac{B}{x+3}\end{aligned}$$

$$\therefore 13x-6 = A(x+3)+B(2x-3)$$

Let $x=-3$,

$$\begin{aligned}13(-3)-6 &= B\left(2(-3)-3\right) \\ B &= 5\end{aligned}$$

Let $x=\frac{3}{2}$,

$$\begin{aligned}13\left(\frac{3}{2}\right)-6 &= A\left[\left(\frac{3}{2}\right)+3\right] \\ A &= 3\end{aligned}$$

$$\therefore \frac{13x-6}{2x^2+3x-9} = \mathbf{\frac{3}{2x-3}+\frac{5}{x+3}}$$

(b)

$$\begin{aligned}\int \frac{17x-3}{2x^2+3x-9}\,dx &= \int \left(\frac{13x-6}{2x^2+3x-9}+\frac{4x+3}{2x^2+3x-9}\right)\,dx \\ &= \int \frac{3}{2x-3}\,dx+\int \frac{5}{x+3}\,dx+\int \frac{4x+3}{2x^2+3x+9}\,dx \\ &= \boldsymbol{\frac{3}{2}\ln|2x-3|+5\ln|x+3|+\ln\left|2x^2+3x-9\right|+c}\end{aligned}$$

2. The following is an improper fraction

$$\frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2}$$

By Long Division,

$$\begin{array}{r|l} & x - 2 \\ x^3 - 3x^2 & \overline{x^4 - 5x^3 + 6x^2 - 18} \\ & -\underline{(x^4 - 3x^3)} \\ & -2x^3 + 6x^2 - 18 \\ & -\underline{(-2x^3 + 6x^2)} \\ & \underline{-18} \end{array}$$

$$\begin{aligned} \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2} &= (x-2) - \frac{18}{x^3 - 3x^2} \\ &= (x-2) - \frac{18}{x^2(x-3)} \end{aligned}$$

By partial fractions,

$$\begin{aligned} \frac{18}{x^2(x-3)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3} \\ 18 &= Ax(x-3) + B(x-3) + Cx^2 \end{aligned}$$

Let $x = 0$,

$$\begin{aligned} 18 &= B(0-3) \\ B &= -6 \end{aligned}$$

Let $x = 3$,

$$\begin{aligned} 18 &= C(3)^2 \\ C &= 2 \end{aligned}$$

Let $x = 1$,

$$\begin{aligned} 18 &= A(1)((1)-3) - 6((1)-3) + 2(1)^2 \\ A &= -2 \end{aligned}$$

$$\begin{aligned} \therefore \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2} &= (x-2) - \left(-\frac{2}{x} - \frac{6}{x^2} + \frac{2}{x-3}\right) \\ &= \boldsymbol{x - 2 + \frac{2}{x} + \frac{6}{x^2} - \frac{2}{x-3}} \end{aligned}$$

3.

$$\frac{x-4}{(2x-1)(x+1)^2} = \frac{A}{2x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x-4 = A(x+1)^2 + B(2x-1)(x+1) + C(2x-1)$$

Let $x = -1$,

$$\begin{aligned} -1-4 &= C(2(-1)-1) \\ C &= 1 \end{aligned}$$

Let $x = \frac{1}{2}$,

$$\begin{aligned} \frac{1}{2} - 4 &= A\left(\frac{1}{2}+1\right)^2 \\ A &= -\frac{14}{9} \end{aligned}$$

Let $x = 0$,

$$\begin{aligned} 0-4 &= \left(-\frac{14}{9}\right)(0+1)^2 + B(2(0)-1)((0)+1) + (2(0)-1) \\ B &= \frac{31}{9} \end{aligned}$$

$$\therefore \boldsymbol{\frac{x-4}{(2x-1)(x+1)^2} = -\frac{14}{9(2x-1)} + \frac{31}{9(x+1)} + \frac{C}{(x+1)^2}}$$

4. (a)

$$
\begin{aligned}
9x^3 - 6x^2 + x &= x\left(9x^2 - 6x + 1\right) \\
&= x(3x-1)^2
\end{aligned}
$$

$$\therefore \frac{2x^2 - 3x + 1}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$2x^2 - 3x + 1 = A(3x-1)^2 + Bx(3x-1) + Cx$$

Let $x = 0$,

$$
\begin{aligned}
2(0)^2 - 3(0) + 1 &= A(3(0)-1)^2 \\
A &= 1
\end{aligned}
$$

Let $x = \frac{1}{3}$,

$$
\begin{aligned}
2\left(\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right) + 1 &= C\left(\frac{1}{3}\right) \\
C &= \frac{2}{3}
\end{aligned}
$$

Let $x = 1$,

$$
\begin{aligned}
2(1)^2 - 3(1) + 1 &= (3(1)-1)^2 + B(1)(3(1)-1) + \frac{2}{3}(1) \\
B &= -\frac{7}{3}
\end{aligned}
$$

$$\therefore \boldsymbol{\frac{2x^2 - 3x + 1}{x(3x-1)^2} = \frac{1}{x} - \frac{7}{3(3x-1)} + \frac{2}{3(3x-1)}}$$

(b)

$$
\begin{aligned}
\int \frac{2x^2 - 3x + 1}{x(3x-1)^2}\ dx &= \int \left(\frac{1}{x} - \frac{7}{3(3x-1)} + \frac{2}{3(3x-1)}\right)\ dx \\
&= \ln x - \frac{7}{3}\left(\frac{1}{3}\ln(3x-1)\right) + \frac{2}{3}\left(\frac{(3x-1)^{-1}}{3(-1)}\right) + c \\
&= \boldsymbol{\ln x - \frac{7}{9}\ln(3x-1) - \frac{2}{9(3x-1)} + c}
\end{aligned}
$$

5 Binomial Theorem

5.1 Full Solutions

1. (a)

$$\begin{aligned}(1+px)^6 &= 1 + \binom{6}{1} px + \binom{6}{2} (px)^2 + ... \\ &= \mathbf{1 + 6px + 15p^2x^2 + ...}\end{aligned}$$

(b)

$$\begin{aligned}(1+px)^6(1+qx) &= \left(1 + 6px + 15p^2x^2 + ...\right)(1+qx) \\ &= 1 + (6p+q)x + \left(6pq + 15p^2\right)x^2 + ...\end{aligned}$$

Since the first 2 non-zero terms are 1 and $-\frac{7}{3}x^2$, the coefficient of x is 0

$$\begin{aligned}6p + q &= 0 \\ q &= -6p \text{ ......(1)}\end{aligned}$$

Coefficient of x^2 is $-\frac{7}{3}$

$$6pq + 15p^2 = -\frac{7}{3} \text{ ......(2)}$$

Substitute Equation (1) into Equation (2),

$$\begin{aligned}6p\left(-6p\right) + 15p^2 &= -\frac{7}{3} \\ p^2 &= \frac{1}{9} \\ p &= \pm\frac{1}{3}\end{aligned}$$

Substitute $p = \pm\frac{1}{3}$ into Equation (1),

$$\begin{aligned}q &= -6\left(\pm\frac{1}{3}\right) \\ &= \pm 2\end{aligned}$$

$$\therefore \boldsymbol{p = \pm\frac{1}{3} \qquad q = \pm 2}$$

2. (a)

$$T_{r+1} = \binom{9}{r} \left(x^2\right)^{9-r} \left(-\frac{3}{x}\right)^r$$
$$= \binom{9}{r} (-3)^r \, x^{18-3r}$$

For the x^3 term

$$18 - 3r = 3$$
$$r = 5$$

$$\therefore \text{Coefficient } p = \binom{9}{5} (-3)^5$$
$$= -30618$$

For the x^6 term

$$18 - 3r = 6$$
$$r = 4$$

$$\therefore \text{Coefficient } q = \binom{9}{4} (-3)^4$$
$$= 10206$$

$$\therefore \frac{p}{q} = \frac{-30618}{10206}$$
$$= \mathbf{-3}$$

(b) (i)

$$\begin{aligned}\left(2+\frac{x}{2}\right)^5 &= 2^5 + \binom{5}{1}(2)^{5-1}\left(\frac{x}{2}\right) + \binom{5}{2}(2)^{5-2}\left(\frac{x}{2}\right)^2 \\ &= \mathbf{32 + 40\boldsymbol{x} + 20\boldsymbol{x}^2 + ...}\end{aligned}$$

(ii)

$$\begin{aligned}(1-kx)^2\left(2+\frac{x}{2}\right)^5 &= \left(1-2kx+k^2x^2\right)\left(32+40x+20x^2+...\right) \\ &= ... + (1)\left(20x^2\right) + (-2kx)(40x) + \left(k^2x^2\right)(32) + ... \\ &= ... + \left(32k^2 - 80k + 20\right)x^2\end{aligned}$$

Since the coefficient is -12,

$$\begin{aligned}32k^2 - 80k + 20 &= -12 \\ 2k^2 - 5k + 2 &= 0 \\ (2k-1)(k-2) &= 0\end{aligned}$$

$$\therefore \boldsymbol{k = \frac{1}{2}} \quad \textbf{or} \quad \boldsymbol{k = 2}$$

3. (a)

$$\left(1-\frac{x}{2}\right)^9 = 1^9 + \binom{9}{1}\left(-\frac{x}{2}\right)^1 + \binom{9}{2}\left(-\frac{x}{2}\right)^2 + \binom{9}{3}\left(-\frac{x}{2}\right)^3 + \binom{9}{4}\left(-\frac{x}{2}\right)^4 + ...$$
$$= \mathbf{1 - \frac{9}{2}\boldsymbol{x} + 9\boldsymbol{x}^2 - \frac{21}{2}\boldsymbol{x}^3 + \frac{63}{8}\boldsymbol{x}^4 + ...}$$

(b)

$$\left(4-\frac{1}{x}+\frac{a}{x^2}\right)\left(1-\frac{x}{2}\right)^9$$
$$= \left(4-\frac{1}{x}+\frac{a}{x^2}\right)\left(1-\frac{9}{2}x+9x^2-\frac{21}{2}x^3+\frac{63}{8}x^4+...\right)$$
$$= ... + 4\left(9x^2\right) + \left(-\frac{21}{2}x^3\right)\left(-\frac{1}{x}\right) + \left(\frac{a}{x^2}\right)\left(\frac{63}{8}x^4\right) + ...$$
$$= ... + \left(\frac{372+63a}{8}\right)x^2 + ...$$

Comparing coefficients,

$$54\frac{3}{8} = \frac{372+63a}{8}$$
$$372+63a = 435$$
$$a = \mathbf{1}$$

(c)

$$\left(1-\frac{1}{2}x-x^2\right)^9 = \left(1-\frac{x+2x^2}{2}\right)^9$$

Hence, comparing with part (a),

$$x_{(a)} = x_{(b)} + 2x_{(b)}^2$$

$$\left(1-\frac{1}{2}x-x^2\right)^9 = 1-\frac{9}{2}\left(x+2x^2\right)+9\left(x+2x^2\right)^2-\frac{21}{2}\left(x+2x^2\right)^3+...$$
$$= 1-\frac{9}{2}\left(x+2x^2\right)+9\left(x^2+4x^3+...\right)-\frac{21}{2}\left(x^3+...\right)+...$$
$$= 1-\frac{9}{2}x-9x^2+9x^2+36x^3-\frac{21}{2}x^3+...$$
$$= \mathbf{1-\frac{9}{2}\boldsymbol{x}+\frac{51}{2}\boldsymbol{x}^3+...}$$

4. (a)

$$\begin{aligned}\left(1-\frac{x}{3}\right)^n &= 1^n + \binom{n}{1}(1)^{n-1}\left(-\frac{x}{3}\right)^1 + \binom{n}{2}(1)^{n-2}\left(-\frac{x}{3}\right)^2 + ...\\ &= 1 - \frac{n}{3}x + \frac{n(n-1)}{18}x^2 + ...\end{aligned}$$

(b)

$$\begin{aligned}&\left(2+px+\frac{5}{2}x^2\right)\left(1-\frac{x}{3}\right)^n\\ &= \left(2+px+\frac{5}{2}x^2\right)\left(1-\frac{n}{3}x+\frac{n(n-1)}{18}x^2+...\right)\\ &= 2 + (px)(1) + (2)\left(-\frac{n}{3}x\right) + 2\left(\frac{n(n-1)}{18}x^2\right) + (px)\left(-\frac{n}{3}x\right) + \frac{5}{2}x^2 + ...\\ &= 2 + \left(p - \frac{2n}{3}\right)x + \left[\frac{n(n-1)}{9} - \frac{pn}{3} + \frac{5}{2}\right]x^2 + ...\end{aligned}$$

(c) Given that

$$\begin{aligned}\left(2+px+\frac{5}{2}x^2\right)\left(1-\frac{x}{3}\right)^n &= 2 + \frac{31p}{3}x + \frac{25}{3}x^2 + ...\\ 2 + \left(p-\frac{2n}{3}\right)x + \left[\frac{n(n-1)}{9} - \frac{pn}{3} + \frac{5}{2}\right]x^2 + ... &= 2 + \frac{31p}{3}x + \frac{25}{3}x^2 + ...\end{aligned}$$

Comparing coefficients,

$$\begin{aligned}p - \frac{2n}{3} &= \frac{31p}{3}\\ -\frac{28}{3}p &= \frac{2n}{3}\\ \therefore -28p &= 2n\\ p &= -\frac{1}{14}n \text{ ......(1)}\end{aligned}$$

$$\begin{aligned}\frac{n(n-1)}{9} - \frac{pn}{3} + \frac{5}{2} &= \frac{25}{3}\\ \frac{n(n-1)}{9} - \frac{pn}{3} - \frac{35}{6} &= 0\\ 2n(n-1) - 6pn - 105 &= 0\end{aligned}$$

Substitute Equation (1) into Equation (2),

$$\begin{aligned} 2n^2 - 2n - 6\left(-\frac{1}{14}n\right)n - 105 &= 0 \\ \frac{17}{7}n^2 - 2n - 105 &= 0 \\ 17n^2 - 14n - 735 &= 0 \\ (17n + 105)(n - 7) &= 0 \end{aligned}$$

$$\therefore n = -\frac{105}{17} \text{ (rej)} \qquad \boldsymbol{n = 7}$$

Substitute $n = 7$ into Equation (1),

$$\begin{aligned} p &= -\frac{1}{14}(7) \\ \boldsymbol{p} &\boldsymbol{= -\frac{1}{2}} \end{aligned}$$

(d) With the new values of n and p,

$$\begin{aligned} &\left(2 - \frac{1}{2}x + \frac{5}{2}x^2\right)\left(1 - \frac{x}{3}\right)^7 \\ &= \left(2 - \frac{1}{2}x + \frac{5}{2}x^2\right)\left(\ldots - \frac{7}{3}x + \frac{42}{18}x^2 - \binom{7}{3}(1)^{7-3}\left(-\frac{x}{3}\right)^3 + \ldots\right) \\ &= \left(2 - \frac{1}{2}x + \frac{5}{2}x^2\right)\left(\ldots - \frac{7}{3}x + \frac{7}{3}x^2 - \frac{35}{27}x^3 + \ldots\right) \\ &= \ldots + \left[2\left(-\frac{35}{27}\right) + \left(-\frac{1}{2}\right)\left(\frac{7}{3}\right) + \left(\frac{5}{2}\right)\left(-\frac{7}{3}\right)\right]x^3 + \ldots \\ &= \ldots - \frac{259}{27}x^3 + \ldots \end{aligned}$$

Hence, the coefficient is $\boldsymbol{-\frac{259}{27}}$

Note that for part (d), we only need to find the coefficients from x to x^3 as these are the only terms that will be multiplied to $\left(2 + px + \frac{5}{2}x^2\right)$ to get an x^3 term

6 Exponential & Logarithms

6.1 Full Solutions

1. (a) When $t = 0$

$$\begin{aligned} V &= 45000e^{-k(0)} \\ &= \$45000 \end{aligned}$$

(b) When $t = 11$, $V = \$36300$

$$\begin{aligned} 36300 &= 45000e^{-k(11)} \\ e^{-11k} &= \frac{121}{150} \\ -11k &= \ln\left(\frac{121}{150}\right) \\ k &= -\frac{1}{11}\ln\left(\frac{121}{150}\right) \end{aligned}$$

$$\therefore V = 45000e^{\frac{1}{11}\ln\left(\frac{121}{150}\right)t}$$

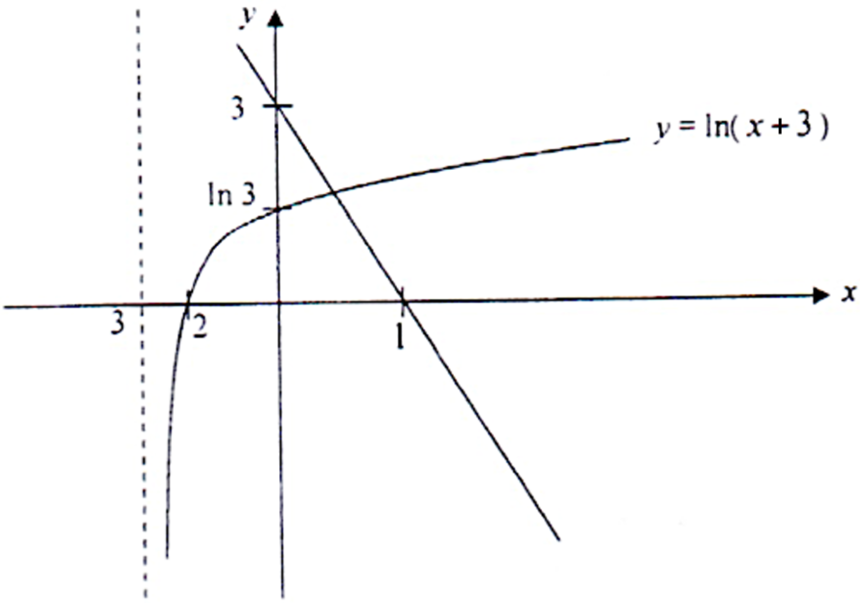
When $t = 9$,

$$\begin{aligned} V &= 45000e^{\frac{1}{11}\ln\left(\frac{121}{150}\right)(9)} \\ &= 37746.03446... \\ &= \mathbf{\$37700\ (nearest\ \$100)} \end{aligned}$$

(c) Since the apartment when it reached $\frac{2}{3}$ of its original value

$$\begin{aligned} \frac{2}{3} &= e^{\frac{1}{11}\ln\left(\frac{121}{150}\right)t} \\ \ln\left(\frac{2}{3}\right) &= \frac{1}{11}\ln\left(\frac{121}{150}\right)t \\ \therefore t &= \frac{\ln\left(\frac{2}{3}\right)}{\frac{1}{11}\ln\left(\frac{121}{150}\right)} \\ &= 20.759717... \\ &\approx \mathbf{21\ (nearest\ month)} \end{aligned}$$

2. (a) Graph for part (a) & (b)



(b)

$$\begin{aligned} ex &= e^{4-3x} - 3e \\ x &= \frac{e^{4-3x}}{e} - 3 \\ e^{3-3x} &= x + 3 \\ \therefore 3 - 3x &= \ln(x+3) \end{aligned}$$

Sketch the graph of: $\boldsymbol{y = 3 - 3x}$

3. (a)

$$5^{x+2} - 25^{x+\frac{1}{2}} = 2\left(5^{x+1}\right)$$
$$\left(5^x\right)\left(5^2\right) - \left(5^{2x}\right)(5) = 2\left(5^x\right)(5)$$

Let $u = 5^x$

$$25u - 5u^2 = 10u$$
$$5u(3-u) = 0$$
$$u = 0 \quad \text{or} \quad u = 3$$
$$5^x = 0 \text{ (rej)} \quad \text{or} \quad 5^x = 3$$

For $5^x = 3$,

$$5^x = 3$$
$$x = \frac{\lg 3}{\lg 5}$$
$$= 0.682606...$$
$$= \mathbf{0.68\ (2.d.p.)}$$

(b)

$$64^x \div 8^y = 32 \text{ ......(1)}$$
$$27^{2x}\left(\frac{1}{\sqrt{3}}\right)^{y+1} = 9\sqrt{3} \text{ ......(2)}$$

From Equation (1),

$$2^{6x} \div 2^{3y} = 2^5$$
$$2^{6x-3y} = 2^5$$
$$\therefore 6x - 3y = 5 \text{ ......(3)}$$

From Equation (2),

$$3^{6x}\left(3^{-\frac{1}{2}(y+1)}\right) = 3^{2\frac{1}{2}}$$
$$\therefore 6x - \frac{1}{2}(y+1) = 2\frac{1}{2}$$
$$y = 12x - 6 \text{ ......(4)}$$

Substitute Equation (4) into Equation (3),

$$6x - 3(12x-6) = 5$$
$$-30x = -13$$
$$x = \frac{13}{30}$$

Substitute $x = \frac{13}{30}$ into Equation (4),

$$y = 12\left(\frac{13}{30}\right) - 6$$
$$= -\frac{4}{5}$$
$$\therefore \boldsymbol{x = \frac{13}{30} \qquad y = -\frac{4}{5}}$$

4. (a)

$$\begin{aligned}
\log_x \frac{p}{\sqrt{q}} - 3\log_x \sqrt{q} &= \log_x (p-q) \\
\log_x \frac{p}{q^{\frac{1}{2}}} - \log_x q^{\frac{3}{2}} &= \log_x (p-q) \\
\log_x \left(\frac{p}{\left(q^{\frac{1}{2}}\right)\left(q^{\frac{3}{2}}\right)} \right) &= \log_x (p-q) \\
\frac{p}{q^2} &= p-q \\
p &= pq^2 - q^3 \\
p\left(q^2-1\right) &= q^3 \\
\therefore \boldsymbol{p} &= \boldsymbol{\frac{q^3}{q^2-1}}
\end{aligned}$$

(b)

$$\begin{aligned}
\log_2 21 + \log_4 \frac{16}{7} &= \log_2 (3\times 7) + \frac{\log_2\left(\frac{16}{7}\right)}{\log_2 4} \\
&= \log_2 3 + \log_2 7 + \frac{1}{2}\left[\log_2 16 - \log_2 7\right] \\
&= \log_2 3 + \log_2 6 + 2 - \frac{1}{2}\log_2 7 \\
&= \log_2 3 + 2 + \frac{1}{2}\log_2 7 \\
&= \boldsymbol{a + 2 + \frac{1}{2}b}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{\left(\sqrt[10]{x}+1\right)\left(x^{\frac{21}{10}} - x^2\right)}{\sqrt[5]{x}-1} &= \frac{\left(\sqrt[10]{x}+1\right)\left(x^2\right)\left(x^{\frac{1}{10}}-1\right)}{\sqrt[5]{x}-1} \\
&= \frac{\left[\left(\sqrt[10]{x}\right)^2 - 1\right]\left(x^2\right)}{\sqrt[5]{x}-1} \\
&= \frac{\left(\sqrt[5]{x}-1\right)\left(x^2\right)}{\sqrt[5]{x}-1} \\
&= \boldsymbol{x^2}
\end{aligned}$$

7 Trigonometry

7.1 Full Solutions

1. (a)

$$\begin{aligned}\tan(\theta - 45^\circ) &= \frac{\tan\theta - \tan 45^\circ}{1 + \tan\theta \tan 45^\circ} \\ &= \frac{\tan\theta - 1}{1 + \tan\theta}\end{aligned}$$

(b)

$$\begin{aligned}\cot 15^\circ &= \cot(60^\circ - 45^\circ) \\ &= \frac{1}{\tan(60^\circ - 45^\circ)} \\ &= \frac{1 + \tan 60^\circ}{\tan 60^\circ - 1} \\ &= \left(\frac{1 + \sqrt{3}}{\sqrt{3} - 1}\right)\left(\frac{\sqrt{3} + 1}{\sqrt{3} + 1}\right) \\ &= \frac{4 + 2\sqrt{3}}{2} \\ &= \mathbf{2 + \sqrt{3}}\end{aligned}$$

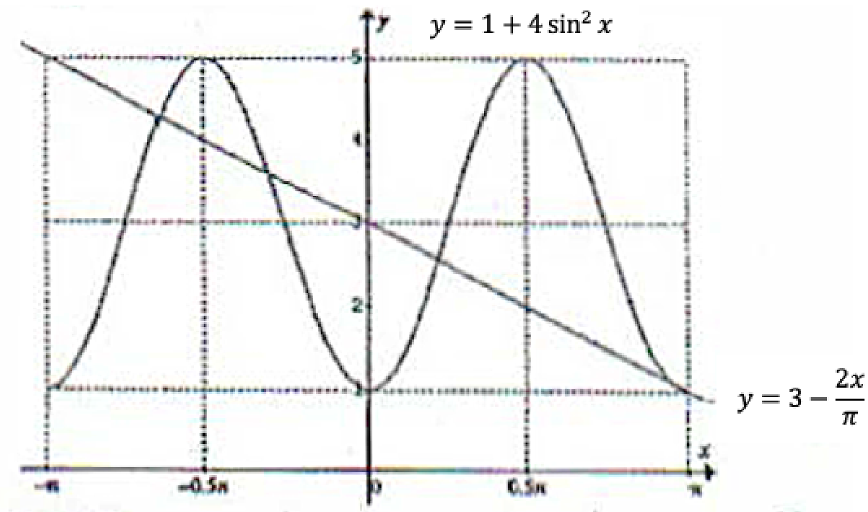
2. (a) (i)

$$\begin{aligned} \text{LHS} &= 1 + 4\sin^2 x \\ &= 1 + 2(1 - \cos 2x) \\ &= 3 - 2\cos 2x \\ &= \text{RHS} \ \textbf{(shown)} \end{aligned}$$

(ii)

Amplitude = **2** Period = $\boldsymbol{\pi}$

(b) Graph for part (b) & (c)



(c)

$$\begin{aligned} \pi \cos 2x &= x \\ \cos 2x &= \frac{x}{\pi} \\ 2\cos 2x &= \frac{2x}{\pi} \\ 3 - 2\cos 2x &= 3 - \frac{2x}{\pi} \end{aligned}$$

Sketch the line: $\boldsymbol{y = 3 - \frac{2x}{\pi}}$

Number of solutions= **5**

3. (a) (i)

$$\begin{aligned}\angle BAC &= 2\pi - \frac{2\pi}{3} - \left(\frac{\pi}{2} - \theta\right) - \frac{\pi}{2} \\ &= \theta + \frac{\pi}{3} \textbf{ (shown)}\end{aligned}$$

(ii)

$$\begin{aligned}\sin\left(\theta + \frac{\pi}{3}\right) &= \frac{BC}{2} \\ BC &= 2\sin\left(\theta + \frac{\pi}{3}\right)\end{aligned}$$

$$\begin{aligned}h &= CD + BC \\ &= \sin\theta + 2\sin\left(\theta + \frac{\pi}{3}\right)\end{aligned}$$

(b)

$$\begin{aligned}h &= \sin\theta + 2\sin\left(\theta + \frac{\pi}{3}\right) \\ &= \sin\theta + 2\sin\theta\cos\frac{\pi}{3} + 2\cos\theta\sin\frac{\pi}{3} \\ &= 2\sin\theta + \sqrt{3}\cos\theta \textbf{ (shown)}\end{aligned}$$

(c)

$$\begin{aligned}R &= \sqrt{2^2 + \left(\sqrt{3}\right)^2} \\ &= \sqrt{7}\end{aligned}$$

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) \\ &= 0.713724... \\ &= 0.714 \text{ (3.s.f.)}\end{aligned}$$

$$\therefore 2\sin\theta + \sqrt{3}\cos\theta = \sqrt{7}\sin(\theta + 0.714)$$

(d)

Maximum value of $h = \boldsymbol{\sqrt{7}}$

(e) Given that $h = 2.5$,

$$2.5 = \sqrt{7}\sin(\theta + 0.714)$$

$$\sin(\theta + 0.714) = \frac{5}{2\sqrt{7}}$$

$$\alpha = \sin^{-1}\left(\frac{5}{2\sqrt{7}}\right) \text{ (Quadrant 1 \& 2)}$$

For Quadrant 1,

$$\begin{aligned}\alpha &= \sin^{-1}\left(\frac{5}{2\sqrt{7}}\right) - \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) \\ &= 0.523598... \\ &= \mathbf{0.524\ (3.s.f.)}\end{aligned}$$

For Quadrant 2,

$$\begin{aligned}\alpha &= \pi - \sin^{-1}\left(\frac{5}{2\sqrt{7}}\right) - \tan^{-1}\left(\frac{\sqrt{3}}{2}\right) \\ &= 1.190545... \\ &= \mathbf{1.19\ (3.s.f.)}\end{aligned}$$

4. (a)

$$\begin{aligned} \text{RHS} &= \sec^2 x \tan^2 x - \sec^2 x + 1 \\ &= \left(\tan^2 x + 1\right)\left(\tan^2 x\right) - \left(\sec^2 x - 1\right) \\ &= \tan^4 x + \tan^2 x - \tan^2 x \\ &= \tan^4 x \\ &= \text{ RHS } \textbf{(shown)} \end{aligned}$$

(b)

$$\begin{aligned} \cos^2 x + 3\sin x \cos x + 1 &= 0 \\ \cos^2 x + 3\sin x \cos x + \sin^2 x + \cos^2 x &= 0 \\ 2\cos^2 x + 3\sin x \cos x + \sin^2 x &= 0 \\ (2\cos x + \sin x)(\cos x + \sin x) &= 0 \end{aligned}$$

$$\therefore \tan x = -2 \qquad \tan x = -1$$

For $\tan x = -2$,

$$\begin{aligned} \alpha &= \tan^{-1}(2) \text{ (Quadrant 2 \& 4)} \\ x &= 180^\circ - \tan^{-1}(2) \\ &= 116.565051... \\ &= \mathbf{116.6^\circ \ (1.d.p.)} \end{aligned}$$

$$\begin{aligned} x &= 360^\circ - \tan^{-1}(2) \\ &= 296.565051... \\ &= \mathbf{296.6^\circ \ (1.d.p.)} \end{aligned}$$

For $\tan x = -1$,

$$\begin{aligned} \alpha &= \tan^{-1}(1) \text{ (Quadrant 2 \& 4)} \\ x &= 180^\circ - \tan^{-1}(1) \\ &= \mathbf{135^\circ} \end{aligned}$$

$$\begin{aligned} x &= 360^\circ - \tan^{-1}(1) \\ &= \mathbf{315^\circ} \end{aligned}$$

(c) (i)

$$\begin{aligned}\sin\theta &= \frac{\sqrt{\left(2\sqrt{2}\right)^2 - \left(\sqrt{3}+1\right)^2}}{2\sqrt{2}}\\ &= \frac{\sqrt{8-\left[3+1+2\sqrt{3}\right]}}{2\sqrt{2}}\\ &= \frac{\sqrt{8-4-2\sqrt{3}}}{2\sqrt{2}}\\ &= \frac{\sqrt{2\left(2-\sqrt{3}\right)}}{2\sqrt{2}}\\ &= \frac{\sqrt{2}\left(\sqrt{2-\sqrt{3}}\right)}{2\sqrt{2}}\\ &= \boldsymbol{\frac{\sqrt{2-\sqrt{3}}}{2}}\end{aligned}$$

(ii)

$$\begin{aligned}\tan\theta &= \frac{\sqrt{4-2\sqrt{3}}}{\sqrt{3}+1}\\ \tan^2\theta &= \left(\frac{\sqrt{4-2\sqrt{3}}}{\sqrt{3}+1}\right)^2\\ &= \frac{4-2\sqrt{3}}{4+2\sqrt{3}}\\ &= \frac{2-\sqrt{3}}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}\\ &= \frac{4+4\sqrt{3}+3}{4-3}\\ &= 7-4\sqrt{3}\end{aligned}$$

$$\therefore \boldsymbol{\tan\theta = \sqrt{7-4\sqrt{3}}} \quad \textbf{(shown)}$$

8 Coordinate Geometry

8.1 Full Solutions

1. (a) (i)

$$\begin{aligned}\text{Gradient of } BE &= \frac{11-8}{8-6} \\ &= \frac{3}{2}\end{aligned}$$

$$\begin{aligned}\therefore \text{Gradient of } AC &= \frac{-1}{\left(\frac{3}{2}\right)} \\ &= -\frac{2}{3}\end{aligned}$$

$$\therefore y-8 = -\frac{2}{3}(x-6)$$

$$\boldsymbol{y = -\frac{2}{3}x + 12}$$

(ii)

$$\boldsymbol{A(0, 12)}$$

(iii) Let the coordinates of F be $F(h, k)$

By similar triangles,

$$\begin{aligned}\frac{8-h}{8-6} &= \frac{3}{1} \\ h &= 2\end{aligned}$$

$$\begin{aligned}\frac{11-k}{11-8} &= \frac{3}{1} \\ k &= 2\end{aligned}$$

$$\therefore \boldsymbol{F(2, 2)}$$

(b)

$$\begin{aligned}\text{Length of } AB &= \sqrt{8^2+1^2} \\ &= \sqrt{65}\end{aligned}$$

$$\begin{aligned}\text{Length of } AP &= \sqrt{4^2+7^2} \\ &= \sqrt{65}\end{aligned}$$

Since $AB = AP$, $\triangle ABP$ is an isosceles triangle **(shown)**

□

Quadrilateral $ABCP$ is a **kite**

(c)

$$\frac{\text{Area of } \triangle ABC}{\text{Area of trapezium } ABCD} = \boldsymbol{\frac{1}{5}}$$

2. (a) For coordinate R, by inspection,

$$\therefore \boldsymbol{R(8,2)}$$

$$\begin{aligned}\text{Gradient of } PR &= \frac{2-4}{8-2} \\ &= -\frac{1}{3}\end{aligned}$$

$$\begin{aligned}\therefore \text{Gradient of } MS &= \frac{-1}{\left(-\frac{1}{3}\right)} \\ &= 3\end{aligned}$$

$$\therefore y-3 = 3(x-5)$$

$$QS : y = 3x - 12 \text{ ......(1)}$$

At S, $y=0$,

$$\begin{aligned}0 &= 3x - 12 \\ x &= 4\end{aligned}$$

$$\therefore \boldsymbol{S(4,0)}$$

$$PQ : y = x + 2 \text{ ......(2)}$$

Let Equation (1) = Equation (2),

$$\begin{aligned}3x - 12 &= x + 2 \\ x &= 7\end{aligned}$$

Substitute $x = 7$ into Equation (2),

$$\begin{aligned}y &= 7 + 2 \\ &= 9\end{aligned}$$

$$\therefore \boldsymbol{Q(7,9)}$$

(b)

$$\begin{aligned}\text{Area of } PQRS &= \frac{1}{2}\begin{vmatrix}4 & 2 & 7 & 8 & 4 \\ 0 & 4 & 9 & 2 & 0\end{vmatrix} \\ &= \frac{1}{2}\left|(48 - (108)\right| \\ &= \frac{1}{2}\left|-60\right| \\ &= \textbf{30 units}^{\mathbf{2}}\end{aligned}$$

3.

$$\begin{aligned} M &= \left(\frac{-5+3}{2}, \frac{6+10}{2}\right) \\ &= (-1, 8) \end{aligned}$$

$$\begin{aligned} \text{Gradient of } AB &= \frac{10-6}{3-(-5)} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \therefore \text{Gradient of perpendicular bisector } MP &= \frac{-1}{\left(\frac{1}{2}\right)} \\ &= -2 \end{aligned}$$

For BP,

$$\begin{aligned} 6y + 7x &= 0 \\ y &= -\frac{7}{6}x \end{aligned}$$

$$\begin{aligned} \therefore y - 6 &= -\frac{7}{6}(x+5) \\ BP : y &= -\frac{7}{6}x + \frac{1}{6} \text{ ......(1)} \end{aligned}$$

For MP,

$$\begin{aligned} \therefore y - 8 &= -2(x+1) \\ MP : y &= -2x + 6 \text{ ......(2)} \end{aligned}$$

At point P, let Equation (1) = Equation (2),

$$\begin{aligned} -\frac{7}{6}x + \frac{1}{6} &= -2x + 6 \\ \frac{5}{6}x &= \frac{35}{6} \\ x &= 7 \end{aligned}$$

Substitute $x = 7$ into Equation (2),

$$\begin{aligned} y &= -2(7) + 6 \\ &= -8 \end{aligned}$$

$$\therefore \boldsymbol{P(7, -8)}$$

4. (a)

$$2y = -4x + 1$$
$$y = -2x + \frac{1}{2}$$
$$\therefore \text{Gradient of } BC = -2$$
$$y - 7 = -2(x - 2)$$
$$BC : \boldsymbol{y = -2x + 11}$$

(b) At F, $y = 0$

$$0 = -2x + 11$$
$$x = -5\frac{1}{2}$$
$$F\left(-5\frac{1}{2}, 0\right)$$

$$\text{Gradient of } AB = \frac{7-(-2)}{2-(-4)}$$
$$= \frac{3}{2}$$

$$\therefore y - 7 = \frac{3}{2}(x - 2)$$
$$AB : y = \frac{3}{2}x + 4$$
$$\therefore E(0, 4)$$

$$\text{Gradient of } EF = \frac{0-4}{-5\frac{1}{2}-0}$$
$$= \frac{8}{11}$$

$$\text{Product of gradients } = \frac{3}{2} \times \frac{8}{11}$$
$$= \frac{12}{11} \neq -1$$

$\therefore$ ***EF* is not perpendicular to *AB***

(c) Let the coordinates of C be (x, y)

$$BC : y = -2x + 11 \text{ ......(1)}$$

Since $AC = AE$,

$$\begin{aligned}\sqrt{(-4-x)^2 + (-2-y)^2} &= \sqrt{(0-x)^2 + (4-y)^2}\\ 16 + 8x + x^2 + 4 + 4y + y^2 &= x^2 + 16 - 8y + y^2\\ 8x + 12y &= -4\\ 2x + 3y &= -1 \text{ ......(2)}\end{aligned}$$

Substitute Equation (1) into Equation (2),

$$\begin{aligned}2x + 3(-2x + 11) &= -1\\ 2x - 6x + 33 &= -1\\ -4x &= -34\\ x &= 8\frac{1}{2}\end{aligned}$$

Substitute $x = 8\frac{1}{2}$ into Equation (1),

$$\begin{aligned}y &= -2\left(8\frac{1}{2}\right) + 11\\ &= -6\end{aligned}$$

$$\therefore \boldsymbol{C\left(8\frac{1}{2}, -6\right)}$$

(d)

$$\begin{aligned}\text{Area of } \triangle AEC &= \frac{1}{2}\begin{vmatrix}-4 & 0 & 8\frac{1}{2} & -4\\ -2 & 4 & -6 & -2\end{vmatrix}\\ &= \frac{1}{2}\left|(-33) - (58)\right|\\ &= \frac{1}{2}\left|-91\right|\\ &= \mathbf{45\frac{1}{2}\ units^2}\end{aligned}$$

9 Further Coordinate Geometry

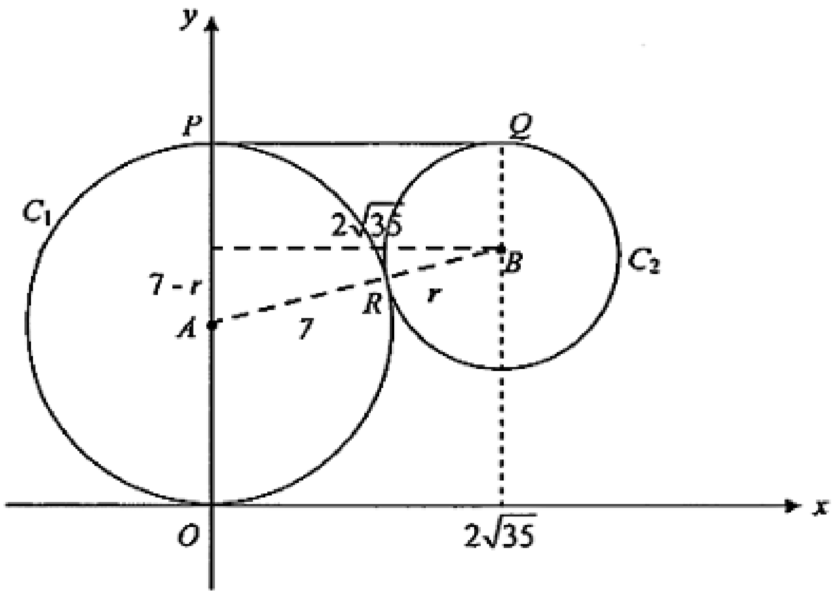
9.1 Full Solutions

1. (a)

$$x^2 + y^2 - 14y = 0$$
$$x^2 + y^2 - 14y + 49 = 49$$
$$(x-0)^2 + (y-7)^2 = 7^2$$

$\therefore$ **Centre = $\boldsymbol{A(0,7)}$ Radius = 7 units**

(b) Add the additional lines as shown below



By Pythagoras' Theorem,

$$(7-r)^2 + \left(2\sqrt{35}\right)^2 = (7+r)^2$$
$$49 - 14r + r^2 + 140 = 49 + 14r + r^2$$
$$r = 5$$

$$\therefore B\left(2\sqrt{35}, 9\right)$$

$$\left(x - 2\sqrt{35}\right)^2 + (y-9)^2 = 5^2$$

$$\boldsymbol{\left(x - 2\sqrt{35}\right)^2 + (y-9)^2 = 25}$$

(c)

$$\text{Midpoint of } AB = \left(\frac{0+2\sqrt{35}}{2}, \frac{7+9}{2}\right)$$
$$= \left(\sqrt{35}, 8\right)$$

$$\text{Gradient of } AB = \frac{9-7}{2\sqrt{35}-0}$$
$$= \frac{1}{\sqrt{35}}$$

$$\text{Gradient of perpendicular bisector} = \frac{-1}{\left(\frac{1}{\sqrt{35}}\right)}$$
$$= -\sqrt{35}$$

$$y - 8 = -\sqrt{35}\left(x - \sqrt{35}\right)$$

$$\therefore \boldsymbol{y = -\sqrt{35}x + 43}$$

2. (a)

Radius = 3 units

$$(x-2)^2 + (y+1)^2 = 3^2$$

$$\boldsymbol{x^2 + y^2 - 4x + 2y - 4 = 0}$$

(b)

$$\text{Gradient of perpendicular bisector} = -\frac{1}{5}$$

Since the perpendicular bisector cuts the centre of the circle,

$$y - (-1) = -\frac{1}{5}(x-2)$$

$$\boldsymbol{y = -\frac{1}{5}x - \frac{3}{5}}$$

(c)

$$\boldsymbol{C(-8, -1)}$$

3. (a) Since $AF : FB = 1 : 2$, by proportion

$$y\text{-coordinate of } A = \frac{\left(1\frac{1}{2}\right)}{2} \times 3$$
$$= 2\frac{1}{4}$$
$$\therefore A\left(0, 2\frac{1}{4}\right)$$

(b)

$$\text{Radius of } C_2 = \sqrt{\left(-\frac{1}{2} - \left(-1\frac{1}{2}\right)\right)^2 + \left(1\frac{1}{2} - 0\right)^2}$$
$$= \sqrt{(1)^2 + \left(1\frac{1}{2}\right)^2}$$
$$= \sqrt{\frac{13}{4}}$$
$$= \frac{\sqrt{13}}{2} \text{ units}$$

$$\text{Equation of } C_2 : \left(x - \left(-1\frac{1}{2}\right)\right)^2 + (y - 0)^2 = \left(\frac{\sqrt{13}}{2}\right)^2$$
$$\therefore \boldsymbol{\left(x + 1\frac{1}{2}\right)^2 + y^2 = \frac{13}{4}}$$

(c) A point that the perpendicular bisector will cut is the midpoint of PF

$$\text{Midpoint of } PF = \left(\frac{-\frac{1}{2} + 0}{2}, \frac{1\frac{1}{2} + (-1)}{2}\right)$$
$$= \left(-\frac{1}{4}, \frac{1}{4}\right)$$

To find the gradient of the perpendicular bisector, we first need to find the gradient of PF first.

$$\text{Gradient of } PF = \frac{-1\frac{1}{2} - (-1)}{-\frac{1}{2} - 0}$$
$$= -5$$

$$\therefore \text{Gradient of perpendicular bisector} = \frac{1}{5}$$

$$\text{Equation: } y - \frac{1}{4} = \frac{1}{5}\left[x - \left(-\frac{1}{4}\right)\right]$$
$$\boldsymbol{y = \frac{1}{5}x + \frac{3}{10}}$$

(d) The y-coordinate of the centre corresponds to the midpoint of P and Q

$$\begin{aligned} y-\text{coordinate of } C_3 &= \frac{2+(-1)}{2} \\ &= \frac{1}{2} \end{aligned}$$

The centre also lies on the perpendicular bisector of PF. Substitute $y = \frac{1}{2}$ into the equation of the perpendicular bisector of PF,

$$\begin{aligned} \frac{1}{2} &= \frac{1}{5}x + \frac{3}{10} \\ x &= 1 \end{aligned}$$

$$\therefore C\left(1, \frac{1}{2}\right)$$

$$\begin{aligned} \text{Radius of } C_3 &= \sqrt{(0-1)^2 + \left(-1-\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{13}{4}} \\ &= \frac{\sqrt{13}}{2} \text{ units} \end{aligned}$$

$$\text{Equation of } C_3 : (x-1)^2 + \left(y-\frac{1}{2}\right)^2 = \left(\frac{\sqrt{13}}{2}\right)^2$$

$$\boldsymbol{\therefore (x-1)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{13}{4}}$$

4. (a)

$$\begin{aligned}\text{Gradient of line} &= \frac{2-0}{-2-(-4)} \\ &= 1\end{aligned}$$

$$\therefore y = x + 4 \text{ ......(1)}$$

$$x^2 + y^2 + 3x - y = 0 \text{ ......(2)}$$

Substitute Equation (1) into Equation (2),

$$\begin{aligned}x^2 + (x+4)^2 + 3x - (x+4) &= 0 \\ x^2 + x^2 + 8x + 16 + 3x - x - 4 &= 0 \\ 2x^2 + 10x + 12 &= 0 \\ x^2 + 5x + 6 &= 0 \\ (x+2)(x+3) &= 0\end{aligned}$$

$$x = -2 \text{ (N.A.)} \qquad x = -3$$

Substitute $x = -3$ into Equation (1),

$$\begin{aligned}y &= -3 + 4 \\ &= 1\end{aligned}$$

$$\therefore \boldsymbol{Q(-3, 1)}$$

(b)

$$\begin{aligned}\text{Midpoint of } PQ &= \left(\frac{-2-3}{2}, \frac{2+1}{2}\right) \\ &= \left(-2\frac{1}{2}, 1\frac{1}{2}\right)\end{aligned}$$

$$\text{Gradient of perpendicular bisector} = -1$$

$$\therefore y - 1\frac{1}{2} = -\left(x - \left(-2\frac{1}{2}\right)\right)$$

$$\boldsymbol{y = -x - 1}$$

(c)

$$x^2 + y^2 + 3x - y = 0$$
$$x^2 + y^2 + 2\left(\frac{3}{2}\right)x + 2\left(-\frac{1}{2}\right)y = 0$$

$$\therefore \text{Radius } = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$
$$= \sqrt{\frac{5}{2}}$$

Let the new centre be (a, b)

$$C_2 : (x-a)^2 + (y-b)^2 = \left(\sqrt{\frac{5}{2}}\right)^2$$

$$C_2 : (x-a)^2 + (y-b)^2 = \frac{5}{2} \text{ ......(1)}$$

The perpendicular bisector of PQ will intersect the centre of C_2

$$b = -a - 1 \text{ ......(2)}$$

Substitute Equation (2) into Equation (1),

$$(x-a)^2 + (y-(-a-1))^2 = \frac{5}{2}$$

Since the circle passes through $P(-2, 2)$,

$$(-2-a)^2 + (3+a)^2 = \frac{5}{2}$$
$$4 + 4a + a^2 + 9 + 6a + a^2 - \frac{5}{2} = 0$$
$$2a^2 + 10a + 10\frac{1}{2} = 0$$
$$4a^2 + 20a + 21 = 0$$
$$(2a+3)(2a+7) = 0$$

$$a = -\frac{3}{2} \text{ (N.A.)} \qquad a = -3\frac{1}{2}$$

Substitute $a = -\frac{7}{2}$ into Equation (2)

$$b = -\left(-\frac{7}{2}\right) - 1$$
$$= 2\frac{1}{2}$$

$$\therefore \boldsymbol{\left(x + 3\frac{1}{2}\right)^2 + \left(y - 2\frac{1}{2}\right) = 2\frac{1}{2}}$$

10 Linear Law

10.1 Full Solutions

1. (a)

$$\begin{aligned} y^2 &= e^{-ax+4} \\ 2\ln y &= -ax+4 \\ \ln y &= -\frac{a}{2}(x)+2 \end{aligned}$$

Using $(4,-4)$

$$\begin{aligned} -4 &= -\frac{a}{2}(4)+2 \\ a &= 3 \end{aligned}$$

At $(2,b)$

$$\begin{aligned} b &= -\frac{3}{2}(2)+2 \\ &= -1 \end{aligned}$$

$$\therefore \boldsymbol{a=3} \qquad \boldsymbol{b=-1}$$

(b) When $x=2$,

$$\begin{aligned} y^2 &= e^{-3(2)+4} \\ y &= \pm\sqrt{e^{-2}} \text{ (rej -ve)} \\ &= 0.367879... \\ &= \mathbf{0.368\ (3.s.f.)} \end{aligned}$$

2. (a)

$$\begin{aligned} y &= ax^{b+1} \\ \lg y &= \lg\left[ax^{b+1}\right] \\ \lg y &= \lg a + \lg x^{b+1} \\ \lg y &= (b+1)\lg x + \lg a \\ Y &= mX + c \end{aligned}$$

Plot a graph of $\lg y$ against $\lg x$ with $(b+1)$ as the gradient and $\lg a$ as the Y-intercept

$\lg x$	0.30	0.48	0.60	0.70	0.78	0.88
$\lg y$	0.75	1.02	1.20	1.35	1.47	1.61

Graph is drawn on the next page

(b)

$$\begin{aligned} \text{Gradient} &= \frac{1.5-0.9}{0.8-0.4} \\ (b+1) &= 1.5 \\ \therefore \boldsymbol{b} &\boldsymbol{= 0.5} \end{aligned}$$

$$\begin{aligned} Y\text{-intercept} &= 0.3 \\ \lg a &= 0.3 \\ \therefore a &= 10^{0.3} \\ &= 1.995262... \\ &= \mathbf{2.00\ (3.s.f.)} \end{aligned}$$

(c)

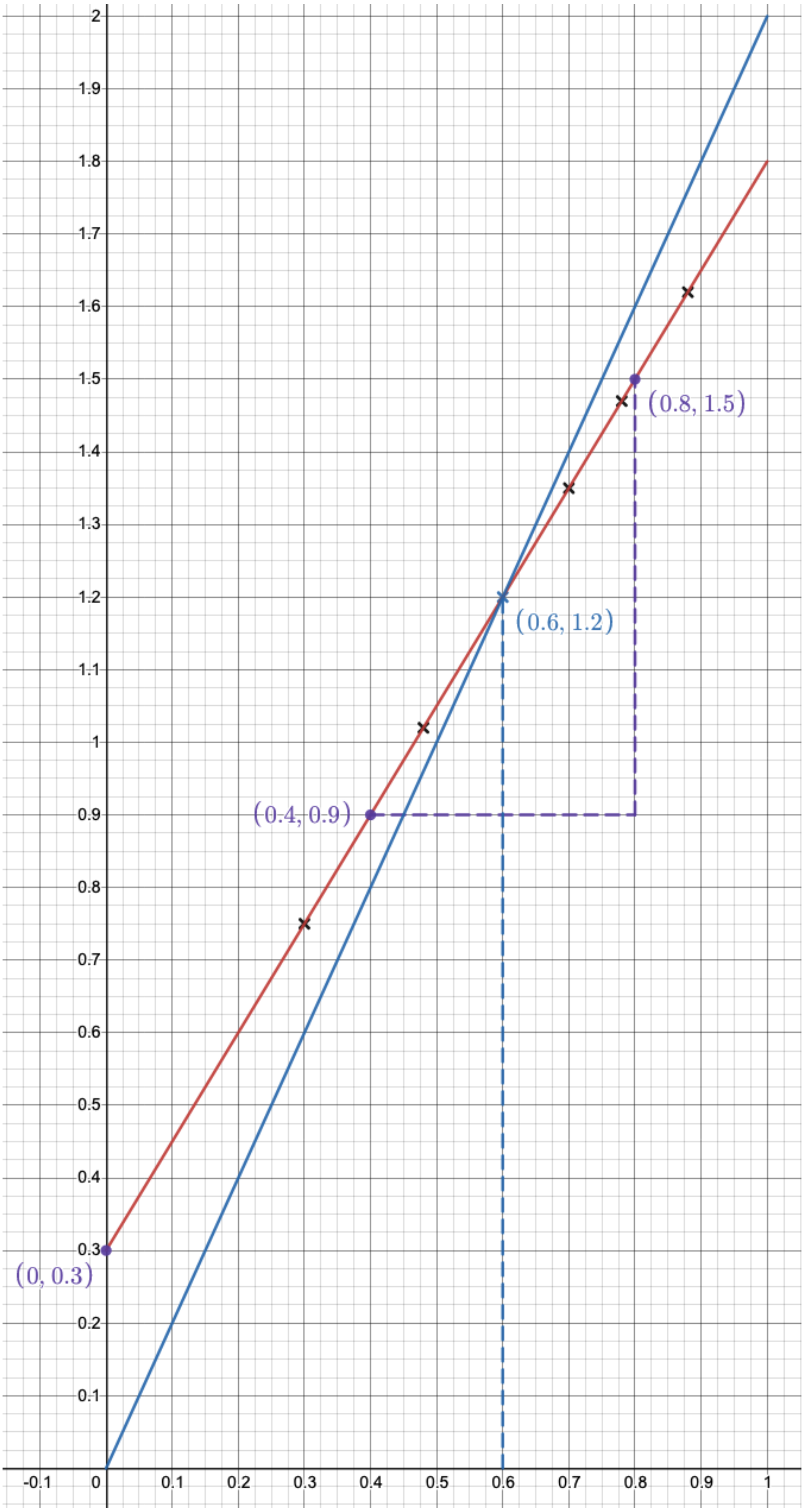
$$\begin{aligned} y &= x^2 \\ \lg y &= 2\lg x \\ Y &= mX + c \end{aligned}$$

Plot the line of $\lg y = 2\lg x$

$$\begin{aligned} x^{1-b} &= a \\ \lg x^{1-b} &= \lg a \\ (1-b)\lg x &= \lg a \\ \lg x - b\lg x &= \lg a \\ \lg x + \lg x &= \lg a + b\lg x + \lg x \\ 2\lg x &= (b+1)\lg x + \lg a \end{aligned}$$

Hence, we are looking for the points of intersection of the 2 lines

$$\begin{aligned} \therefore \lg x &= 0.6 \\ x &= 10^{0.6} \\ &= 3.981071... \\ &= \mathbf{3.98\ (3.s.f.)} \end{aligned}$$


(0.8, 1.5)
(0.6, 1.2)
(0.4, 0.9)
(0, 0.3)

3. (a)

$$y = p(x+5)^{\frac{3}{2}} - q\sqrt{x+5}$$
$$\frac{y}{\sqrt{x+5}} = p(x+5) - q$$
$$\frac{y}{\sqrt{x+5}} = px + (5p - q)$$
$$Y = mX + c$$

Plot a graph of $\frac{y}{\sqrt{x+5}}$ against x with p as the gradient and $(5p-q)$ as the Y-intercept

x	0.5	1	1.5	2	2.5
$\frac{y}{\sqrt{x+5}}$	10.49	11.51	12.51	13.49	14.50

Graph is drawn on the next page

(b)

$$\text{Gradient} = \frac{14-13}{2.26-1.75}$$
$$\therefore \boldsymbol{p = 1.96}$$

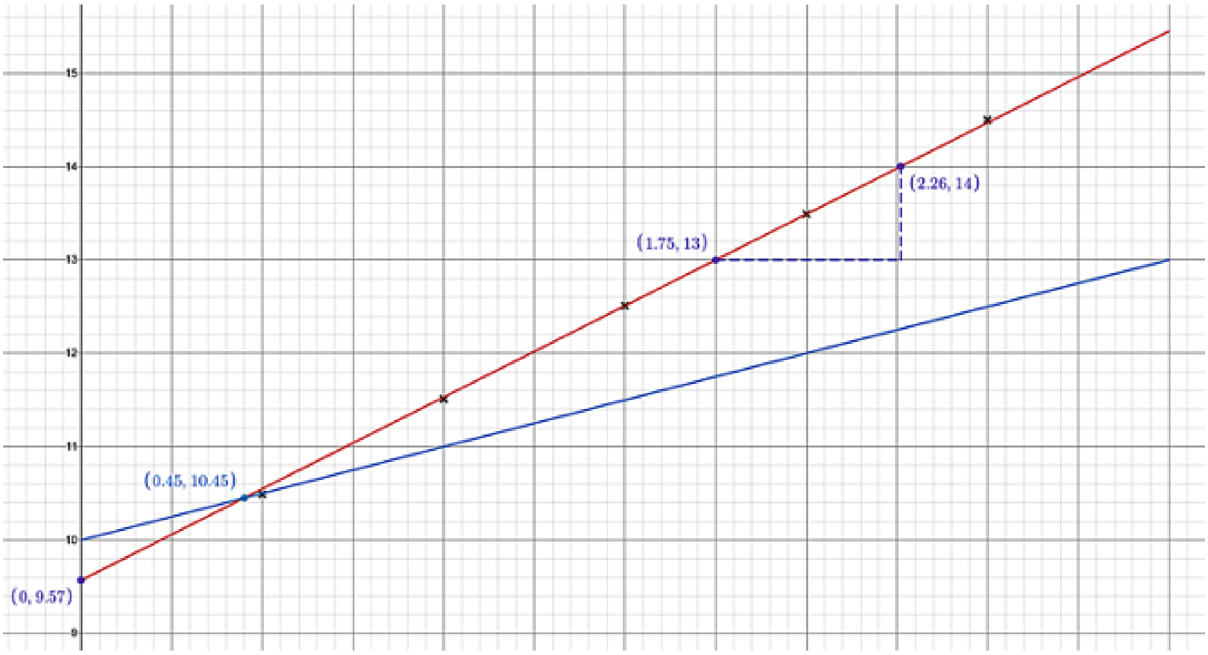
$$Y\text{-intercept} = 9.57$$
$$5p - q = 9.57$$
$$\therefore q = 5(1.96) - 9.57$$
$$= \mathbf{0.23}$$

(c)

$$p(x+5)^{\frac{3}{2}} = \sqrt{x+5}(x+10+q)$$
$$p(x+5) = x + 10 + q$$
$$p(x+5) - q = x + 10$$

Plot the line of $\frac{y}{\sqrt{x+5}} = x + 10$. Hence, we are looking for the point of intersection of the 2 lines

$$\therefore x = \mathbf{0.45}$$



15
14
13
12
11
10
9
(2.26, 14)
(1.75, 13)
(0.45, 10.45)
(0, 9.57)

4. (a)

$$\begin{aligned} y &= \frac{p-x}{x+q} \\ y(x+q) &= p-x \\ x(1+y) &= -qy+p \end{aligned}$$

$$\therefore q = 1\frac{1}{3}$$

Substitute $(3, 2)$,

$$\begin{aligned} 2 &= -1\frac{1}{3}(3)+p \\ p &= 6 \end{aligned}$$

$$\therefore \boldsymbol{p = 6} \qquad \boldsymbol{q = 1\frac{1}{3}}$$

(b)

$$(y, x(1+y)) = (6, k)$$

$$x(1+6) = k$$

$$\therefore \boldsymbol{x = \frac{k}{7}}$$

11 Proofs of Plane Geometry

11.1 Full Solutions

1. (a)

$$\angle FAD = \angle BCD \text{ (angles in the same segment) (A)}$$
$$FD = BD \text{ (given) (S)}$$
$$\angle ADF = \angle CDB \text{ (vertically opposite angles) (A)}$$

By ASA congruency test, $\triangle ADF$ is congruent to $\triangle CDB$

(b)

$$\angle GEA = \angle CEB \text{ (common angle) (A)}$$
$$\angle AGE = \angle CBE \text{ (exterior angles of a cyclic quadrilateral) (A)}$$

By AA similarity test, $\triangle GEA$ is similar to $\triangle BEC$

(c)

$$\begin{aligned} GA : AF &= GA : CB \text{ (corresponding sides of congruent triangles)} \\ &= AE : BE \text{ (ratio of corresponding sides of similar triangles)} \\ &= \mathbf{3 : 1} \end{aligned}$$

(d) **Not in syllabus**

$$\begin{aligned} EH^2 &= EB \times EA \text{ (tangent-secant theorem)} \\ &= EB \times 3EB \\ &= 3EB^2 \textbf{ (proven)} \end{aligned}$$

2. (a)

$$\angle GEC = \angle GCB \text{ (alternate segment theorem) (A)}$$

$$\angle EGC = \angle CGB \text{ (common angle) (A)}$$

By AA similarity test, $\triangle EGC$ is similar to $\triangle CGB$

(b)

$$\angle BCE = \angle GCB \text{ (} BC \text{ bisects } \angle ACE\text{)}$$

$$\angle GEC = \angle GCB \text{ (alternate segment theorem)}$$

$$\therefore \angle BCE = \angle GEC$$

$$\triangle BCE \text{ is an isosceles triangle}$$

$$\therefore BC = BE \textbf{ (proven)}$$

(c) **Not in syllabus**

$$\begin{aligned} GC^2 &= GB \times GE \text{ (tangent-secant theorem)} \\ &= GB \times (GB + BE) \\ &= GB^2 + GB \times BE \\ &= GB^2 + GB \times BC \quad (\because BE = BC) \end{aligned}$$

$$\therefore GC^2 - GB^2 = GB \times BC \textbf{ (proven)}$$

(d) **Not in syllabus**

$$\begin{aligned} DG \times GB &= AG \times GC \text{ (intersecting chord theorem)} \\ \frac{DG}{AG} &= \frac{GC}{GB} \end{aligned}$$

$$\begin{aligned} \left(\frac{DG}{AG}\right)^2 &= \left(\frac{GC}{GB}\right)^2 \\ &= \frac{(GC)^2}{(GB)^2} \\ &= \frac{GB \times GE}{(GB)^2} \text{ (tangent-secant theorem)} \\ &= \frac{GE}{GB} \textbf{ (proven)} \end{aligned}$$

3. (a)

$$\angle TPS = \angle SRP \text{ (alternate segment theorem) (A)}$$
$$\angle SRP = \angle SPR \text{ (RS=PS)}$$
$$\therefore \angle TPS = \angle SPR \textbf{ (proven)}$$

(b)

$$\angle SPT = \angle PQT \text{ (alternate segment theorem) (A)}$$
$$\angle PTS \text{ is a common angle (A)}$$

By AA similarity test, $\triangle SPT$ is similar to $\triangle PQT$

(c) Since $\triangle SPT$ is similar to $\triangle PQT$

$$\frac{SP}{PQ} = \frac{PT}{QT} = \frac{ST}{PT}$$

$$\frac{SP}{PQ} = \frac{PT}{QT}$$
$$PT \times PQ = QT \times SP$$

Since $SP = SR$ (given),

$$\therefore PT \times PQ = QT \times SR \textbf{ (proven)}$$

4. (a)

$$\angle ADG = 90^\circ \text{ (tangent perpendicular to radius)}$$

$$OB \text{ is parallel to } DG \text{ (midpoint theorem)}$$

$$\angle AOB = \angle ADG = 90^\circ \text{ (corresponding angles)}$$

Since OB is the perpendicular bisector of AD

$$AB = DB$$

$$\therefore \boldsymbol{ABD} \textbf{ is an isosceles triangle}$$

(b) By Pythagoras' Theorem,

$$\begin{aligned} AG^2 - DG^2 &= AD^2 \text{ (Pythagoras' Theorem)} \\ (2AB)^2 - (2DF)^2 &= AD^2 \text{ } (AB = BG \text{ and } DF = FG) \\ 4\left(AB^2 - DF^2\right) &= AD^2 \\ 4\left(DB^2 - DF^2\right) &= AD^2 \text{ } (AB = DB) \end{aligned}$$

$$\therefore DB^2 - DF^2 = \frac{1}{4}AD^2 \textbf{ (proven)}$$

(c) In $\triangle ADF$ and $\triangle DCF$,

$$\angle DAF = \angle CDF \text{ (alternate segment theorem) (A)}$$

$$\angle AFD = \angle DFC \text{ (common angles) (A)}$$

By AA similarity test, $\triangle ADF$ is similar to $\triangle DCF$

(d) Since $\triangle ADF$ and $\triangle DCF$ are similar,

$$\begin{aligned} \frac{DF}{CF} &= \frac{AF}{DF} \\ DF^2 &= AF \times CF \end{aligned}$$

Since $GF = DF$,

$$\therefore GF^2 = AF \times CF \textbf{ (proven)}$$

12 Differentiation

12.1 Full Solutions

1. (a)

$$y = \frac{x+1}{(2x-5)^3}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x-5)^3(1) - (x+1)\left[3(2x-5)^2(2)\right]}{\left[(2x-5)^3\right]^2} \\ &= \frac{(2x-5)^2\left[(2x-5) - 6x - 6\right]}{(2x-5)^6} \\ &= \frac{2x-5-6x-6}{(2x-5)^4} \\ &= \frac{-4x-11}{(2x-5)^4} \textbf{ (shown)}\end{aligned}$$

(b) For y to not be an increasing function, $\frac{dy}{dx} \leq 0$

$$-4x - 11 \leq 0$$

$$\therefore \boldsymbol{x \geq -2\frac{3}{4}}$$

(c) When $x = 3$, $\frac{dy}{dt} = 46$

$$\begin{aligned}\therefore \frac{dx}{dt} &= \left.\frac{dx}{dy}\right|_{x=3} \times \frac{dy}{dt} \\ &= \frac{(2(3)-5)^4}{-4(3)-11} \times (46) \\ &= -2\end{aligned}$$

$$\therefore \text{Rate of decrease } = \textbf{2 units/s}$$

(d)

$$z = y^3$$

$$\therefore \frac{dz}{dy} = 3y^2$$

When $x = 3$, $y = 4$

$$\begin{aligned}\therefore \left.\frac{dz}{dy}\right|_{y=4} &= 3(4)^2 \\ &= 48\end{aligned}$$

$$\begin{aligned}\therefore \frac{dz}{dt} = \frac{dz}{dy} \times \frac{dy}{dt} \\ &= (48)(46) \\ &= \textbf{2208 units/s}\end{aligned}$$

2. (a)

$$\angle DEC = 150^\circ - 90^\circ$$
$$= 60^\circ$$

$\therefore \triangle CDE$ is an equilateral triangle

$$\therefore \text{Perimeter: } 6x + 2y = 4$$

$$\boldsymbol{y = 2 - 3x}$$

$$\begin{aligned}\therefore \text{Area of frame} &= A_{\text{ABCD}} + A_{\triangle\text{CDE}} \\ &= 2xy + \frac{1}{2}(2x)(2x)\sin 60^\circ \\ &= 2x(2-3x) + 2x^2\left(\frac{\sqrt{3}}{2}\right) \\ &= 2x - 6x^2 + \sqrt{3}x^2 \\ &= 4x + \left(\sqrt{3}-6\right)x^2 \ \textbf{(shown)}\end{aligned}$$

(b)

$$\frac{dA}{dx} = 4 + 2\left(\sqrt{3}-6\right)x$$

Since the area of the frame is a maximum, $\frac{dA}{dx} = 0$

$$\therefore 4 + 2\left(\sqrt{3}-6\right)x = 0$$

$$\boldsymbol{x = -\frac{2}{\sqrt{3}-6}}$$

$$\frac{d^2A}{dx^2} = 2\left(\sqrt{3}-6\right) < 0$$

Hence, from the second derivative test, A is maximum

$$\begin{aligned}\therefore \text{Max } A &= 4\left(-\frac{2}{\sqrt{3}-6}\right) + \left(\sqrt{3}-6\right)\left(-\frac{2}{\sqrt{3}-6}\right)^2 \\ &= 0.937218... \\ &= \mathbf{0.937\ (3.s.f.)}\end{aligned}$$

3. (a)

$$\begin{aligned}\text{Total volume} &= 120\\(3x)(3x)(x)+\pi\left(x^2\right)y &= 120\\9x^3+\pi x^2 y &= 120\end{aligned}$$

$$\boldsymbol{y=\frac{120-9x^3}{\pi x^2}}$$

(b)

$$\begin{aligned}\text{Total surface area} &= 2\left(9x^2\right)+4\left(3x^2\right)+2\pi xy\\&= 30x^2+2\pi x\left(\frac{120-9x^3}{\pi x^2}\right)\\&= 30x^2+\frac{240}{x}-18x^2\\\therefore A &= \frac{240}{x}+12x^2 \ \textbf{(shown)}\end{aligned}$$

(c)

$$\begin{aligned}A &= \frac{240}{x}-12x^2\\\frac{dA}{dx} &= -\frac{240}{x^2}+24x\end{aligned}$$

Since the surface area is stationary, $\frac{dA}{dx}=0$

$$\begin{aligned}-\frac{240}{x^2}+24x &= 0\\24x^3 &= 240\\x^3 &= 10\\x &= \sqrt[3]{10}\end{aligned}$$

$$\begin{aligned}\therefore A &= \frac{240}{\sqrt[3]{10}}+12\left(\sqrt[3]{10}\right)^2\\&= 167.097198...\\&= \mathbf{167\ mm^2\ (3.s.f.)}\end{aligned}$$

(d)

$$\frac{d^2A}{dx^2}=\frac{480}{x^3}+24$$

$$\begin{aligned}\therefore \left.\frac{d^2A}{dx^2}\right|_{x=\sqrt[3]{10}} &= \frac{480}{\left(\sqrt[3]{10}\right)^3}+24\\&= 72>0\end{aligned}$$

Hence, the stationary value of A is a **minimum**

4. (a)

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sin x}{2\tan x+\cos x}\right) &= \frac{(2\tan x+\cos x)\cos x-\sin x\left(2\sec^2 x-\sin x\right)}{(2\tan x+\cos x)^2}\\ &= \frac{2\tan x\cos x+\cos^2 x-2\sin x\sec^2 x+\sin^2 x}{(2\tan x+\cos x)^2}\\ &= \frac{2\sin x-2\sin x\left(1+\tan^2 x\right)+\cos^2 x+\sin^2 x}{(2\tan x+\cos x)^2}\\ &= \frac{2\sin x-2\sin x-2\sin x\tan^2 x+1}{(2\tan x+\cos x)^2}\\ &= \frac{1-2\sin x\tan^2 x}{(2\tan x+\cos x)^2}\end{aligned}$$

$$\therefore \boldsymbol{a=1} \qquad \boldsymbol{b=-1}$$

(b)

$$y=(1+x)\,e^{3x}$$

$$\begin{aligned}\frac{dy}{dx} &= (1+x)\,3e^{3x}+(1)e^{3x}\\ &= 4e^{3x}+3xe^{3x}\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= 12e^{3x}+\left[3e^{3x}+3x\left(3e^{3x}\right)\right]\\ &= 15e^{3x}+9xe^{3x}\end{aligned}$$

$$\begin{aligned}\therefore \text{RHS} &= 9y+\frac{d^2y}{dx^2}\\ &= 9\left[(1+x)\,e^{3x}\right]+15e^{3x}+9xe^{3x}\\ &= 9e^{3x}+9xe^{3x}+15e^{3x}+9xe^{3x}\\ &= 24e^{3x}+18xe^{3x}\\ &= 6\left(4e^{3x}+3xe^{3x}\right)\\ &= 6\left(\frac{dy}{dx}\right)\\ &= \text{LHS } \textbf{(shown)}\end{aligned}$$

13 Integration

13.1 Full Solutions

1. Point of intersection between the 2 curves:

$$\frac{54}{x} = 2x^2$$
$$x^3 = 27$$
$$x = 3$$

$$\begin{aligned}\therefore \text{Area of shaded region} &= \int_0^3 2x^2 \ dx + \int_3^7 \frac{54}{x} \ dx \\ &= \left[\frac{2}{3}x^3\right]_0^3 + [54\ln x]_3^7 \\ &= \frac{2}{3}(27) + 54(\ln 7 - \ln 3) \\ &= 63.754084... \\ &= \mathbf{63.8 \ units^2 \ (3.s.f.)}\end{aligned}$$

2. (a) Based on the given information, we can see that $f(x)$ is continuous

$$\begin{aligned}\therefore \int_0^5 f(x) \ dx + \int_5^6 f(x) \ dx &= \int_0^2 f(x) \ dx + \int_2^6 f(x) \ dx \\ &= 10 + 14 \\ &= \mathbf{24}\end{aligned}$$

(b) (i)

$$\begin{aligned}\int \sqrt{2x+1} \ dx &= \int (2x+1)^{\frac{1}{2}} \ dx \\ &= \frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2}(2)} + c \\ &= \boldsymbol{\frac{1}{2}(2x+1)^{\frac{3}{2}} + c}\end{aligned}$$

(ii)

$$\begin{aligned}\int \frac{2x^{\frac{1}{2}}}{x\sqrt{x}} \ dx &= 2\int \frac{1}{x} \ dx \\ &= \boldsymbol{2\ln|x| + c}\end{aligned}$$

3. (a)

$$\begin{aligned}\int_4^8 f(x)\ dx &= \int_0^8 f(x)\ dx - \int_0^4 f(x)\ dx \\ &= 16 - (-7) \\ &= \mathbf{23}\end{aligned}$$

(b)

$$\begin{aligned}\text{Area of shaded region} &= \int_0^4 f(x) + 3\ dx \\ &= \int_0^4 f(x)\ dx + \int_0^4 3\ dx \\ &= (-7) + [3x]_0^4 \\ &= -7 + 12 \\ &= \mathbf{5\ units^2}\end{aligned}$$

4. (a) (i) When $n = 1$,

$$f'(x)\big|_{n=1} = \frac{8}{2x+1}$$

$$\begin{aligned}\therefore f(x) &= \int \frac{8}{2x+1}\,dx \\ &= 4\ln(2x+1) + c\end{aligned}$$

Hence, since $f(1) = 0$,

$$\begin{aligned}4\ln 3 + c &= 0 \\ c &= -4\ln 3\end{aligned}$$

$$\therefore \boldsymbol{f(x) = 4\ln(2x+1) - 4\ln 3} \quad \textbf{OR} \quad \boldsymbol{f(x) = 4\ln\left(\frac{2x+1}{3}\right)}$$

(ii) When $n = 4$,

$$f'(x)\big|_{n=1} = \frac{8}{(2x+1)^4}$$

$$\begin{aligned}\therefore f(x) &= \int \frac{8}{(2x+1)^4}\,dx \\ &= 8\int (2x+1)^{-4}\,dx \\ &= -\frac{4}{3}(2x+1)^{-3} + c\end{aligned}$$

Hence, since $f(1) = 0$,

$$-\frac{4}{3}(2(1)+1)^{-3} + c = 0$$

$$c = \frac{4}{81}$$

$$\therefore \boldsymbol{f(x) = \frac{4}{81} - \frac{4}{3(2x+1)^3}}$$

(b) For $f(x)$ to have any stationary points, $f'(x) = 0$

$$\frac{8}{(2x+1)^n} = 0$$

For the above to be well-defined, $n < 0$

$$\therefore \boldsymbol{n \geq 0}$$

14 Differentiation & Integration

14.1 Full Solutions

1. (a)

$$y = \frac{2x}{\sqrt{8x - x^2}}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\left(\sqrt{2x - x^2}\right)(2) - 2x\left[\frac{1}{2}\left(8x - x^2\right)^{-\frac{1}{2}}(8 - 2x)\right]}{8x - x^2} \\
&= \frac{2\sqrt{8x - x^2} - \frac{8x - 2x^2}{\sqrt{8x - x^2}}}{8x - x^2} \\
&= \frac{2\left(8x - x^2\right) - 8x + 2x^2}{\sqrt{\left(8x - x^2\right)^3}} \\
&= \frac{8x}{\sqrt{\left(8x - x^2\right)^3}} \textbf{ (shown)}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_2^5 \frac{2x}{\sqrt{\left(8x - x^2\right)^3}}\, dx &= \frac{1}{4}\int_2^5 \frac{8x}{\sqrt{\left(8x - x^2\right)^3}}\, dx \\
&= \frac{1}{4}\left[\frac{2x}{\sqrt{8x - x^2}}\right]_2^5 \\
&= \frac{1}{4}\left[\frac{2(5)}{\sqrt{8(5) - (5)^2}} - \frac{2(2)}{\sqrt{8(2) - (2)^2}}\right] \\
&= \frac{1}{4}\left[\frac{10}{\sqrt{15}} - \frac{4}{\sqrt{12}}\right] \\
&= \frac{5}{2\sqrt{3}\sqrt{5}} - \frac{1}{2\sqrt{3}} \\
&= \frac{5 - \sqrt{5}}{2\sqrt{3}\sqrt{5}} \times \frac{\sqrt{3}\sqrt{5}}{\sqrt{3}\sqrt{5}} \\
&= \frac{5\sqrt{3}\sqrt{5} - 5\sqrt{3}}{30} \\
&= \frac{\sqrt{3}}{6}\left(\sqrt{5} - 1\right)
\end{aligned}$$

(c)

$$\begin{aligned} y|_{x=4} &= \frac{2(4)}{\sqrt{8(4)-(4)^2}} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{x=4} &= \frac{8(4)}{\sqrt{(8(4)-(4)^2)^3}} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \therefore \text{Gradient of normal} &= \frac{-1}{\left(\frac{1}{2}\right)} \\ &= -2 \end{aligned}$$

$$\therefore y - 2 = -2(x-4)$$

$$\boldsymbol{y = -2x + 10}$$

2. (a)

$$\frac{d}{dx}\left(xe^{2x}\right) = 2xe^{2x} + e^{2x}$$
$$= \boldsymbol{e^{2x}(2x+1)}$$

(b) At the stationary point, $\frac{dy}{dx} = 0$

$$\therefore e^{2x}(2x+1) = 0$$

$$e^{2x} = 0 \text{ (rej)} \quad \text{or} \quad \boldsymbol{x = -\frac{1}{2}}$$

(c)

$$\begin{aligned}
\int_0^2 4xe^{2x}\ dx &= 2\int_0^2 \left(2xe^{2x} + e^{2x} - e^{2x}\right)\ dx \\
&= 2\int_0^2 2xe^{2x} + e^{2x}\ dx - 2\int_0^2 e^{2x}\ dx \\
&= 2\left[xe^{2x}\right]_0^2 - 2\left[\frac{1}{2}e^{2x}\right]_0^2 \\
&= 2\left((2)e^{2(2)}\right) - 2\left[\frac{1}{2}e^{2(2)} - \frac{1}{2}e^{2(0)}\right] \\
&= 164.794450... \\
&= \textbf{165 (3.s.f.)}
\end{aligned}$$

3. (a)

$$y = \frac{3x^2}{x-1}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x-1)(6x) - \left(3x^2\right)(1)}{(x-1)^2} \\ &= \frac{6x^2 - 6x - 3x^2}{(x-1)^2} \\ &= \mathbf{\frac{3x(x-2)}{(x-1)^2}}\end{aligned}$$

(b)

$$\begin{aligned}\int_2^4 \frac{x^2-2x}{3(x-1)^2}\,dx &= \frac{1}{9}\int_2^4 \frac{3\left(x^2-2x\right)}{(x-1)^2}\,dx \\ &= \frac{1}{9}\left[\frac{3x^2}{x-1}\right]_2^4 \\ &= \frac{1}{9}\left[\frac{3(4)^2}{(4)-1} - \frac{3(2)^2}{(2)-1}\right] \\ &= \mathbf{\frac{4}{9}}\end{aligned}$$

(c) Given that $\frac{dy}{dt} = -4$,

$$\begin{aligned}\left.\frac{dx}{dt}\right|_{x=3} &= \frac{dx}{dy} \times \frac{dy}{dt} \\ &= \left[\frac{((3)-1)^2}{3(3)((3)-2)}\right](-4) \\ &= \mathbf{-1\frac{7}{9}\ \text{units/second}}\end{aligned}$$

4. (a)

$$y = \frac{x^3}{3} + x^2 - 8x$$

$$\frac{dy}{dx} = x^2 + 2x - 8$$

At the stationary points, $\frac{dy}{dx} = 0$

$$\therefore x^2 + 2x - 8 = 0$$
$$(x-2)(x+4) = 0$$

$$\therefore x = 2 \qquad \text{or} \qquad x = -4$$

$$\frac{d^2y}{dx^2} = 2x + 2$$

When $x = 2$,

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 2(2) + 2$$
$$= 6 > 0$$

Hence, $x = 2$ is a **minimum point**

When $x = -4$,

$$\left.\frac{d^2y}{dx^2}\right|_{x=-4} = 2(-4) + 2$$
$$= -6 < 0$$

Hence, $x = -4$ is a **maximum point**

(b)

$$\text{Area under the curve} = \int_a^0 \left(\frac{1}{3}x^3 + x^2 - 8x\right)\, dx - \int_0^b \left(\frac{1}{3}x^3 + x^2 - 8x\right)\, dx$$
$$= \left[\frac{1}{12}x^4 + \frac{1}{3}x^3 - 4x^2\right]_a^0 - \left[\frac{1}{12}x^4 + \frac{1}{3}x^3 - 4x^2\right]_0^b$$
$$= \left[0 - \left(\frac{1}{12}a^4 + \frac{1}{3}a^3 - 4a^2\right)\right] - \left[\left(\frac{1}{12}b^4 + \frac{1}{3}b^3 - 4b^2\right)\right]$$
$$= -\frac{1}{12}a^4 - \frac{1}{3}a^3 + 4a^2 - \frac{1}{12}b^4 - \frac{1}{3}b^3 + 4b^2$$
$$= \left[4\left(a^2 + b^2\right) - \frac{1}{12}\left(a^4 + b^4\right) - \frac{1}{3}\left(a^3 + b^3\right)\right] \text{ square units } \textbf{(shown)}$$

5. (a) Let

$$f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$$

$$\begin{aligned}
f'(x) &= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2} \\
&= \frac{\left(\sin x \cos x - \sin^2 x - \cos^2 x + \sin x \cos x\right) - \left(\sin^2 x + 2 \sin x \cos x + \cos^2 x\right)}{(\sin x - \cos x)^2} \\
&= -\frac{2}{(\sin x - \cos x)^2}
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{d}{dx}\left[\ln f(x)\right] &= \frac{f'(x)}{f(x)} \\
&= -\frac{2}{(\sin x - \cos x)^2} \times \frac{\sin x - \cos x}{\sin x + \cos x} \\
&= -\frac{2}{(\sin x + \cos x)(\sin x - \cos x)} \\
&= -\frac{2}{\sin^2 x - \cos^2 x} \\
&= \frac{2}{\cos 2x} \ \textbf{(shown)}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{1 - 2\sin^2 x} &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\cos 2x} \, dx \\
&= \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2}{\cos 2x} \, dx \\
&= \frac{1}{2} \left[\ln\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= \frac{1}{2} \left[\ln\left(\frac{\sin\frac{\pi}{2} + \cos\frac{\pi}{2}}{\sin\frac{\pi}{2} - \cos\frac{\pi}{2}}\right) - \ln\left(\frac{\sin\frac{\pi}{3} + \cos\frac{\pi}{3}}{\sin\frac{\pi}{3} - \cos\frac{\pi}{3}}\right)\right] \\
&= \frac{1}{2} \left[\ln\left(\frac{1+0}{1-0}\right) - \ln\left(\frac{\frac{\sqrt{3}}{2} + \frac{1}{2}}{\frac{\sqrt{3}}{2} - \frac{1}{2}}\right)\right] \\
&= \frac{1}{2} \left[-\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\right] \\
&= -0.658478... \\
&= \mathbf{-0.658\ (3.s.f.)}
\end{aligned}$$

15 Kinematics

15.1 Full Solutions

1. (a)

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= 6t + k \end{aligned}$$

When $t = 0$, $a = -3$

$$\begin{aligned} -3 &= 6(0) + k \\ k &= -3 \textbf{ (shown)} \end{aligned}$$

(b)

$$v = 3t^2 - 3t$$

When the particle is at instantaneous rest, $v = 0$

$$\begin{aligned} 3t^2 - 3t &= 0 \\ 3t(t-1) &= 0 \end{aligned}$$

$$\therefore \boldsymbol{t = 0} \qquad \textbf{or} \qquad \boldsymbol{t = 1}$$

(c)

$$\begin{aligned} s &= \int v\ dt \\ &= t^3 - \frac{3}{2}t^2 + c \end{aligned}$$

When $t = 0$, $S = 0$, $c = 0$

$$\therefore s = t^3 - \frac{3}{2}t^2$$

When $t = 1$,

$$\begin{aligned} s &= (1)^3 - \frac{3}{2}(1)^2 \\ &= -\frac{1}{2} \end{aligned}$$

When $t = 4$,

$$\begin{aligned} s &= (4)^3 - \frac{3}{2}(4)^2 \\ &= 40 \end{aligned}$$

$$\begin{aligned} \therefore \text{Total distance} &= 40 + \frac{1}{2}(2) \\ &= 41 \text{ m} \end{aligned}$$

$$\begin{aligned} \therefore \text{Average speed} &= \frac{41}{4} \\ &= \boldsymbol{10\frac{1}{4}} \textbf{ m/s} \end{aligned}$$

2. (a) When the particle is at instantaneous rest, $v = 0$

$$\begin{aligned} 5\left(1 - e^{1-t}\right) &= 0 \\ e^{1-t} &= 1 \\ 1 - t &= \ln 1 \\ t &= \mathbf{1} \end{aligned}$$

(b)

$$\begin{aligned} s &= \int v \ dt \\ &= 5t + 5e^{1-t} + c \end{aligned}$$

When $t = 0$, $s = 0$

$$\begin{aligned} \therefore 5e + c &= 0 \\ c &= -5e \end{aligned}$$

$$\therefore s = 5t + 5e^{1-t} - 5e$$

$$\begin{aligned} \therefore \text{Distance} &= s|_{t=2} - s|_{t=1} \\ &= \left[5(2) + 5e^{1-2} - 5e\right] - \left[5(1) + 5e^{1-1} - 5e\right] \\ &= 1.839397... \\ &= \mathbf{1.84\ m\ (3.s.f.)} \end{aligned}$$

(c)

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= 5e^{1-t} \end{aligned}$$

When $t = 2.5$,

$$\begin{aligned} a &= 5e^{1-2.5} \\ &= 1.115650... \\ &= \mathbf{1.12\ m/s^2} \end{aligned}$$

(d) As $t \to \infty$, $e^{1-t} \to 0$

$$\therefore \boldsymbol{v} = \mathbf{5\ m/s}$$

3. (a) When $t = 0$,

$$\begin{aligned} a\big|_{t=0} &= 2\cos\left(\frac{0}{3}\right) \\ &= \mathbf{2\ ms^{-2}} \end{aligned}$$

(b)

$$\begin{aligned} v &= \int a\ dt \\ &= 6\sin\left(\frac{t}{3}\right) + c \end{aligned}$$

When $t = 0$, $v = 2$

$$\begin{aligned} \therefore 2 &= 6\sin 0 + c \\ c &= 2 \end{aligned}$$

$$\therefore v = 6\sin\left(\frac{t}{3}\right) + 2$$

At instantaneous rest, $v = 0$

$$\begin{aligned} 6\sin\left(\frac{t}{3}\right) + 2 &= 0 \\ \sin\left(\frac{t}{3}\right) &= -\frac{1}{3} \end{aligned}$$

$$\alpha = \sin^{-1}\left(\frac{1}{3}\right) \text{ (Quadrant 3 \& 4)}$$

In Quadrant 3,

$$\begin{aligned} \frac{t}{3} &= \pi + \sin^{-1}\left(\frac{1}{3}\right) \\ t &= 3\left[\pi + \sin^{-1}\left(\frac{1}{3}\right)\right] \\ &= 10.444288... \\ &= 10.4 \text{ sec} \end{aligned}$$

In Quadrant 4,

$$\begin{aligned} \frac{t}{3} &= 2\pi - \sin^{-1}\left(\frac{1}{3}\right) \\ t &= 3\left[2\pi - \sin^{-1}\left(\frac{1}{3}\right)\right] \\ &= 17.830045... \\ &= 17.8 \text{ sec} \end{aligned}$$

$$\therefore \boldsymbol{t = 10.4 \textbf{ sec} \quad \textbf{or} \quad t = 17.8 \textbf{ sec}}$$

(c)

$$\begin{aligned} s &= \int v \ dt \\ &= -18\cos\left(\frac{t}{3}\right) + 2t + c \end{aligned}$$

When $t = 0$, $s = 0$

$$\begin{aligned} \therefore 0 &= -18\cos 0 + 2(0) + c \\ c &= 18 \end{aligned}$$

$$\therefore s = -18\cos\left(\frac{t}{3}\right) + 2t + 18$$

When $t = 10.444...$ sec,

$$\begin{aligned} s &= -18\cos\left(\frac{3\left[\pi + \sin^{-1}\left(\frac{1}{3}\right)\right]}{3}\right) + 2\left\{3\left[\pi + \sin^{-1}\left(\frac{1}{3}\right)\right]\right\} + 18 \\ &= 55.859140... \text{ m} \end{aligned}$$

When $t = 15$ sec,

$$\begin{aligned} s &= -18\cos\left(\frac{15}{3}\right) + 2(15) + 18 \\ &= 42.894080... \text{ m} \end{aligned}$$

$$\begin{aligned} \therefore \text{Total distance travelled} &= (55.859140...) + (55.859140... - 42.894080...) \\ &= 68.82... \\ &= \mathbf{68.8 \text{ m } (3.s.f.)} \end{aligned}$$

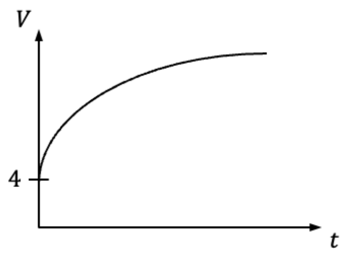
4. (a)

$$\frac{dv}{dt} = 5 - t$$

When the velocity is maximum, $\frac{dv}{dt} = 0$

$$\therefore 0 = 5 - t$$
$$t = 5 \text{ sec } \textbf{(shown)}$$

(b) Velocity-time graph



(c) When $t = 5$,

$$v = 5(5) - \frac{1}{2}(5)^2 + 4$$
$$= 16\frac{1}{2}$$

Since the deceleration is uniform, it will form a straight-line graph with a negative gradient of -1.5

$$\therefore v - 16\frac{1}{2} = -1\frac{1}{2}(t - 5)$$
$$v = -1\frac{1}{2}t + 24$$

Hence, at B, $v = 0$

$$\therefore 0 = -1\frac{1}{2}t + 24$$
$$t = \textbf{16 sec}$$

(d)

$$\begin{aligned}\text{Total distance} &= \text{Area under the graph}\\ &= \int_0^5 5t - \frac{1}{2}t^2 + 4\ dt + \frac{1}{2}\left(16\frac{1}{2}\right)(16-5)\\ &= \left[\frac{5}{2}t^2 - \frac{1}{6}t^3 + 4t\right]_0^5 + 90\frac{3}{4}\\ &= \left[\frac{5}{2}(5)^2 - \frac{1}{6}(5)^3 + 4(5)\right] + 90\frac{3}{4}\\ &= \mathbf{152\frac{5}{12}\ m}\end{aligned}$$