Ong Kai Wen

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June Practice Questions 2022 Full Solutions (A-Math)

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Question Source

All questions are sourced and selected based on the known abilities of students sitting for the 'O' Level A-Math Examination. All questions compiled here are from **2015-2016 School Mid-Year / Prelim Papers**. Questions are categorised into the various topics and range in varying difficulties. If questions are sourced from respective sources, credit will be given when appropriate.

How to read:

[S4 ABCSS P1/2011 PRELIM Qn 1]

Secondary 4, ABC Secondary School, Paper 1, 2011, Prelim, Question 1

Syllabus (4049)

Algebra	**Geometry and Trigonometry**	**Calculus**
Quadratic Equations & Inequalities	Trigonometry	Differentiation
Surds	Coordinate Geometry	Integration
Polynomials	Further Coordinate Geometry	Kinematics
Simultaneous Equations	Linear Law	
Partial Fractions	Proofs of Plane Geometry	
Binomial Theorem		
Exponential & Logarithms		

Contents

1 Quadratic Equations & Inequalities

1.1 Full Solutions

1. (a) (i)

$$y = 4x^2 + px + p - 6 \text{ ......(1)}$$
$$y = -3 \text{ ......(2)}$$

Let Equation (1) = Equation (2),

$$4x^2 + px + p - 6 = -3$$
$$4x^2 + px + p - 3 = 0$$

Since the line intersect the curve, $b^2 - 4ac \geq 0$,

$$(p)^2 - 4(4)(p-3) \geq 0$$
$$p^2 - 16p + 48 \geq 0$$
$$(p-12)(p-4) \geq 0$$

$$\therefore \boldsymbol{p \leq 4} \qquad \textbf{and} \qquad \boldsymbol{p \geq 12}$$

(ii) Since the line ($y = -3$) is tangential to the curve

$$\therefore \boldsymbol{p = 4} \qquad \textbf{or} \qquad \boldsymbol{p = 12}$$

(iii) Since the curve has a positive y-intercept,

$$p - 6 > 0$$
$$\boldsymbol{p > 6}$$

(b) To show that the equation has real and distinct roots, WTS: $b^2 - 4ac > 0$

$$\begin{aligned} b^2 - 4ac &= (4m+3)^2 - 4(m+1)(2m-1) \\ &= 16m^2 + 24m + 9 - 4\left(2m^2 + m - 1\right) \\ &= 16m^2 + 24m + 9 - 8m^2 - 4m + 4 \\ &= 8m^2 + 20m + 13 \\ &= 8\left[m^2 + \frac{5}{2}m\right] + 13 \\ &= 8\left(m + \frac{5}{4}\right)^2 + \frac{1}{2} \end{aligned}$$

For all real values of m,

$$\left(m + \frac{5}{4}\right)^2 \geq 0$$
$$\left(m + \frac{5}{4}\right)^2 + \frac{1}{2} > 0$$
$$b^2 - 4ac > 0$$

Since the discriminant is greater than0, the equation has real and distinct roots for all real values of m

□

2. (a) (i)

$$x^2 + px - x + p^2 + 2 = x^2 + (p-1)x + \left(p^2 + 2\right)$$

To show that the expression is always positive, WTS: $b^2 - 4ac < 0$

$$\begin{aligned} b^2 - 4ac &= (p-1)^2 - 4(1)\left(p^2+2\right)] \\ &= p^2 - 2p + 1 - 4p^2 - 8 \\ &= -3p^2 - 2p + 7 \\ &= -3\left[p^2 + \frac{2}{3}p\right] + 7 \\ &= -3\left(p + \frac{1}{3}\right)^2 - 6\frac{2}{3} \end{aligned}$$

For all real values of p,

$$\begin{aligned} \left(p + \frac{1}{3}\right)^2 &\geq 0 \\ -3\left(p + \frac{1}{3}\right)^2 &\leq 0 \\ -3\left(p + \frac{1}{3}\right)^2 - 6\frac{2}{3} &< 0 \end{aligned}$$

Since the coefficient of x^2 is positive and the discriminant is less than 0, the expression is always positive for all real values of x

□

(ii)

$$\frac{x^2 - 3x - 38}{x^2 + px - x + p^2 + 2} < 0$$

From part (a), since the denominator is always positive,

$$\begin{aligned} x^2 - 3x - 28 &< 0 \\ (x-7)(x+4) &< 0 \end{aligned}$$

$$\therefore \boldsymbol{-4 < x < 7}$$

(b)

$$y = 2x - k \text{ ......(1)}$$

$$y^2 = x + k \text{ ......(2)}$$

Substitute Equation (1) into Equation (2),

$$\begin{aligned} (2x-k)^2 &= x + k \\ 4x^2 - 4xk + k^2 - x - k &= 0 \\ 4x^2 + (-4k-1)x + \left(k^2 - k\right) &= 0 \end{aligned}$$

Since the line cuts the curve at 2 distinct points, $b^2 - 4ac > 0$

$$\begin{aligned} (-4k-1)^2 - 4(4)\left(k^2 - k\right) &> 0 \\ 16k^2 + 8k + 1 - 16k^2 + 16k &> 0 \\ 24k + 1 &> 0 \end{aligned}$$

$$\therefore \boldsymbol{k > -\frac{1}{24}}$$

3. (a)

$$3(2x-5)^2 > x(2x-5)$$

$$\begin{aligned} 3(2x-5)^2 - x(2x-5) &> 0 \\ (2x-5)\left[3(2x-5) - x\right] &> 0 \\ (2x-5)(5x-15) &> 0 \\ 5(2x-5)(x-3) &> 0 \end{aligned}$$

$$\boldsymbol{x < 2\frac{1}{2}} \qquad \textbf{and} \qquad \boldsymbol{x > 3}$$

(b)

$$y = (k+4)x^2 + 4x - k$$

(i) To show that the curve meets the x-axis, WTS: $b^2 - 4ac \geq 0$

$$\begin{aligned} b^2 - 4ac &= (4)^2 - 4(k+4)(-k) \\ &= 16 + 4k^2 + 16k \\ &= 4\left(k^2 + 4k + 4\right) \\ &= 4(k+2)^2 \end{aligned}$$

For all real values of k,

$$\begin{aligned} (k+2)^2 &\geq 0 \\ 4(k+2)^2 &\geq 0 \end{aligned}$$

Since the discriminant is greater than or equals to 0, the curve meets the x-axis

□

(ii) When the x-axis is tangential to the curve, $b^2 - 4ac = 0$

$$\begin{aligned} 4(k+2)^2 &= 0 \\ k &= \mathbf{-2} \end{aligned}$$

(c) Since the following equation is always positive, $b^2 - 4ac < 0$

$$\begin{aligned} (4)^2 - 4(p)(q) &< 0 \\ 4pq &> 16 \\ pq &> 4 \end{aligned}$$

Since the curve is always positive, the coefficient of x^2 must also be positive

$$\therefore \boldsymbol{p > 0} \qquad \textbf{and} \qquad \boldsymbol{pq > 4}$$

4. (a)

$$x^2 + (a-2)x - 2a = 0$$

To show that the roots of the equation are real, $b^2 - 4ac \geq 0$

$$\begin{aligned} b^2 - 4ac &= (a-2)^2 - 4(1)(-2a) \\ &= a^2 - 4a + 4 + 8a \\ &= a^2 + 4a + 4 \\ &= (a+2)^2 \end{aligned}$$

For all real values of a,

$$(a+2) \geq 0$$

Since the discriminant is greater than or equals to 0, the roots of the equation are real

□

(b)

$$y = (b-3)x^2 - 2bx + (b-2)$$

For the curve to always be positive, the coefficient of x^2 is always positive

$$\begin{aligned} b - 3 &> 0 \\ b &> 3 \end{aligned}$$

Since the curve is always positive, $b^2 - 4ac < 0$

$$\begin{aligned} (-2b)^2 - 4(b-3)(b-2) &< 0 \\ 4b^2 - 4\left(b^2 - 5b + 6\right) &< 0 \\ 4b^2 - 4b^2 + 20b - 24 &< 0 \\ 20b &< 24 \\ b &< \frac{6}{5} \end{aligned}$$

Since a condition is that $b > 3$, there are no values of b for which y is always positive

□

2 (Indices) and Surds

2.1 Full Solutions

1.

$$\text{Area of triangle} = \frac{1}{2} \times (\text{Length of triangle}) \times (\text{Height of triangle})$$
$$\left(1 + \frac{5\sqrt{5}}{2}\right) = \frac{1}{2}\left(3 + 2\sqrt{5}\right)(\text{Height of triangle})$$
$$\frac{2 + 5\sqrt{5}}{2} = \left(\frac{3 + 2\sqrt{5}}{2}\right)(\text{Height of triangle})$$

Hence,

$$\begin{aligned} \therefore \text{Height of triangle} &= \frac{2 + 5\sqrt{5}}{3 + 2\sqrt{5}} \times \frac{3 - 2\sqrt{5}}{3 - 2\sqrt{5}} \\ &= \frac{6 - 4\sqrt{5} + 15\sqrt{5} - 50}{-11} \\ &= \frac{11\sqrt{5} - 44}{-11} \\ &= \left(\mathbf{4 - \sqrt{5}}\right) \ \mathbf{cm} \end{aligned}$$

2. (a)

$$\begin{aligned}\text{LHS} &= \frac{2(4)^{\frac{1}{2}x+2} - 2^{x+1}}{6^x \times 3^{1-2x}} \\ &= \frac{2(2)^{x+4} - 2^{x+1}}{2^x \times 3^x \times \left(\dfrac{3}{3^{2x}}\right)} \\ &= \frac{2^x\left(2^5 - 2\right)}{2^x \times \left(\dfrac{3}{3^x}\right)} \\ &= 10\left(3^x\right)\end{aligned}$$

$$\therefore \boldsymbol{k = 10} \qquad \boldsymbol{n = 1}$$

(b) (i)

$$\begin{aligned}\text{Height of } AC &= \frac{7\sqrt{2}}{2\sqrt{2}+1} \times \frac{2\sqrt{2}-1}{2\sqrt{2}-1} \\ &= \frac{28 - 7\sqrt{2}}{7} \\ &= \boldsymbol{\left(4 - \sqrt{2}\right) \text{ cm}}\end{aligned}$$

(ii) By Pythagoras' Theorem,

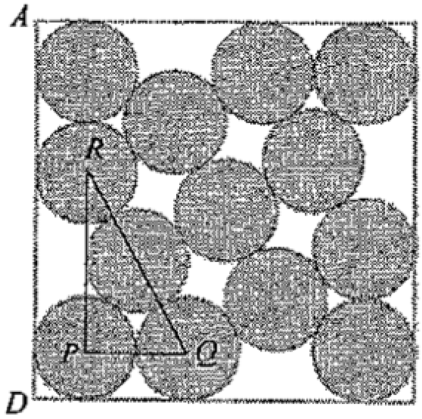
$$\begin{aligned}BC^2 &= AC^2 - AB^2 \\ &= \left(4 - \sqrt{2}\right)^2 - \left(\sqrt{2}+1\right)^2 \\ &= 16 - 8\sqrt{2} + 2 - \left[2 + 2\sqrt{2} + 1\right] \\ &= \boldsymbol{\left(15 - 10\sqrt{2}\right) \text{ cm}}\end{aligned}$$

(iii) Since this is a cuboid of a square base, by Pythagoras' Theorem,

$$\begin{aligned}2\,(\text{Side})^2 &= BC^2 \\ (\text{Side})^2 &= \frac{1}{2}\left(15 - 10\sqrt{2}\right)\end{aligned}$$

$$\begin{aligned}\therefore \text{Volume of cuboid} &= \frac{1}{2}\left(15 - 10\sqrt{2}\right)\left(\sqrt{2}+1\right) \\ &= \frac{1}{2}\left[15\sqrt{2} + 15 - 20 - 10\sqrt{2}\right] \\ &= \frac{1}{2}\left(5\sqrt{2} - 5\right) \\ &= \boldsymbol{\frac{5}{2}\left(\sqrt{2} - 1\right) \text{ cm}^3}\end{aligned}$$

3. (a) Let the points P, Q and R be centres of the circles as shown below



From the diagram above,

$$RQ = 4 \text{ cm} \qquad PQ = 2 \text{ cm}$$

By Pythagoras' Theorem,

$$\begin{aligned} PR &= \sqrt{4^2 - 2^2} \\ &= 2\sqrt{3} \end{aligned}$$

$$\begin{aligned} \therefore AD &= 4 + PR \\ &= \left(\mathbf{4 + 2\sqrt{3}}\right) \ \textbf{cm} \end{aligned}$$

(b)

$$\begin{aligned} \text{Area of square} &= \left(4 + 2\sqrt{3}\right)^2 \\ &= 16 + 16\sqrt{3} + 12 \\ &= \left(\mathbf{28 + 16\sqrt{3}}\right) \ \mathbf{cm^2} \end{aligned}$$

4. (a)

$$
\begin{aligned}
\frac{6\sqrt{5}}{2\sqrt{5}-4} &= \frac{6\sqrt{5}}{2\sqrt{5}-4} \times \frac{2\sqrt{5}+4}{2\sqrt{5}+4} \\
&= \frac{60+24\sqrt{5}}{4} \\
&= \mathbf{15+6\sqrt{5}}
\end{aligned}
$$

(b) (i)

$$
\begin{aligned}
AC^2 &= \left(13\sqrt{5}-1\right)^2 \\
&= 845 - 26\sqrt{5} + 1 \\
&= \mathbf{846 - 26\sqrt{5}}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\text{Volume of pyramid} &= \frac{1}{3} \times AB \times BC \times \text{Height} \\
&= \frac{1}{3} \times AC^2 \times \text{Height} \qquad \text{[Pythagoras' Theorem]} \\
&= \frac{1}{3} \times \left(846 - 26\sqrt{5}\right)\left(15 + 6\sqrt{5}\right) \qquad \text{[part (a)]} \\
&= \frac{1}{3}\left(12690 + 5076\sqrt{5} - 390\sqrt{5} - 780\right) \\
&= \left(\mathbf{3970 + 1562\sqrt{5}}\right) \ \mathbf{cm^3}
\end{aligned}
$$

5. (a) We shall first prove that $\triangle ABC$ is similar to $\triangle AXB$

$$\angle AXB = \angle ABC \text{ (given) (A)}$$

$$\angle BAX \text{ is a common angle (A)}$$

$$\therefore \text{By the } \mathbf{AA} \text{ similarity test, } \triangle ABC \text{ is similar to } \triangle AXB$$

Hence, using the above proof,

$$\frac{AB}{AX} = \frac{AC}{AB}$$

$$AX \times AC = AB^2 \textbf{ (shown)}$$

□

(b)

$$AX \times AC = AB^2$$

$$\begin{aligned} AX &= \frac{AB^2}{AC} \\ &= \frac{\left(2\sqrt{2}-1\right)^2}{\left(4\sqrt{2}+7\right)} \times \frac{4\sqrt{2}-7}{4\sqrt{2}-7} \\ &= \frac{\left(9-4\sqrt{2}\right)\left(4\sqrt{2}-7\right)}{-17} \\ &= \frac{36\sqrt{2}-63-32+28\sqrt{2}}{-17} \\ &= \frac{64\sqrt{2}-95}{-17} \\ &= \boldsymbol{\frac{1}{17}\left(95-64\sqrt{2}\right)} \end{aligned}$$

(c)

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= \left(2\sqrt{2}-1\right)^2 + \left(72+60\sqrt{2}\right) \\ &= 9-4\sqrt{2}+72+60\sqrt{2} \\ &= 81+56\sqrt{2} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{81+56\sqrt{2}} \\ &= \sqrt{\left(4\sqrt{2}+7\right)^2} \\ &= \left(4\sqrt{2}+7\right) \text{ cm} \end{aligned}$$

Hence, this shows that $\triangle AXB$ obeys Pythagoras' Theorem, $\angle AXB = 90°$

□

3 Polynomials

3.1 Full Solutions

1. (a)

$$f(x) = 6x^3 + 3x^2 - x + 2 = 0$$

Let $x = -1$,

$$\begin{aligned} f(-1) &= 6(-1)^3 + 3(-1)^2 - (-1) + 2 \\ &= 0 \end{aligned}$$

$$\therefore (x+1) \text{ is a factor of } f(x)$$

Let a be an arbitrary constant

$$f(x) = (x+1)\left(6x^2 + ax + 2\right)$$

Comparing coefficient of x,

$$\begin{aligned} 2 + a &= -1 \\ a &= -3 \end{aligned}$$

$$f(x) = (x+1)\left(6x^2 - 3x + 2\right)$$

For the quadratic factor,

$$\begin{aligned} b^2 - 4ac &= (-3)^2 - 4(6)(2) \\ &= -39 < 0 \end{aligned}$$

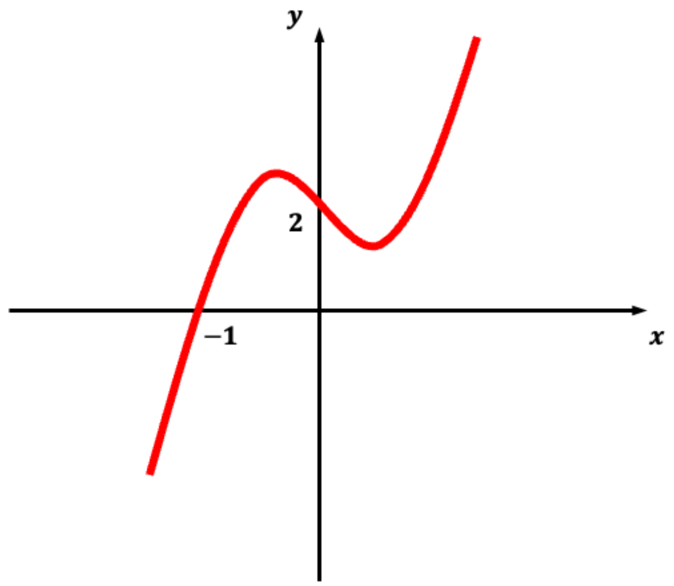
$$\therefore \left(6x^2 - 3x + 2\right) \text{ has no real roots}$$

Hence, there is only one real root

□

$$x = \mathbf{-1}$$

(b) Diagram



2. (a) Since $(x+2)$ is a factor of $f(x)$,

$$\begin{aligned} f(-2) &= 0 \\ f(-2) &= (-2)^3 + p(-2)^2 + q(-2) + 4 \\ 0 &= -8 + 4p - 2q + 4 \end{aligned}$$

$$2p - q = 2 \text{ ......(1)}$$

Since $(x+1)$ leaves a remainder of 6 when it divides $f(x)$,

$$\begin{aligned} f(-1) &= 6 \\ 6 &= (-1)^3 + p(-1)^2 + q(-1) + 4 \end{aligned}$$

$$p = 3 + q \text{ ......(2)}$$

Substitute Equation (2) into Equation (1),

$$\begin{aligned} 2(3+q) - q &= 6 \\ q &= 0 \end{aligned}$$

Hence, substitute $q = 0$ into Equation (2),

$$p = 3$$

$$\therefore p = \mathbf{3} \qquad q = \mathbf{0}$$

(b) Let a be an arbitrary constant

$$f(x) = (x+2)\left(x^2 + ax + 2\right)$$

Comparing coefficient of x,

$$\begin{aligned} 2a + 2 &= 0 \\ a &= -1 \end{aligned}$$

$$f(x) = \boldsymbol{(x+2)\left(x^2 - x + 2\right)}$$

3. (a) Let a be an arbitrary constant

$$\begin{aligned} 4x^3+5x-3 &= (2x-a)\left(2x^2+x+m\right) \\ &= 4x^3+2x^2+2mx-2ax^2-ax-am \\ &= 4x^2+(2-2a)x^2+(2m-a)x-am \end{aligned}$$

Comparing coefficients,

$$\begin{aligned} 2-2a &= 0 \\ a &= 1 \end{aligned}$$

$$\begin{aligned} -3 &= -(1)\,m \\ m &= \mathbf{3} \end{aligned}$$

(b) (i)

$$f(x) = 3(x+2)(x-3)(x-k)$$

Since the remainder of 42 when divided by $(x+1)$,

$$\begin{aligned} f(-1) &= 42 \\ 3(-1+2)(-1-3)(-1-k) &= 42 \\ k &= \mathbf{2\frac{1}{2}} \end{aligned}$$

(ii)

$$f(x) = 3(x+2)(x-3)\left(x-2\frac{1}{2}\right)$$

When divided by x,

$$\begin{aligned} f(0) &= 3(0+2)(0-3)\left(0-2\frac{1}{2}\right) \\ &= \mathbf{45} \end{aligned}$$

4. (a) When $f(x)$ is divided by $(2x-1)$ and leaves a remainder of $1\frac{3}{8}$

$$f\left(\frac{1}{2}\right) = 1\frac{3}{8}$$
$$1 + 2\left(\frac{1}{2}\right) + A\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 = 1\frac{3}{8}$$
$$A = \mathbf{-2}$$

(b)

$$f(x) = -x^3 - 2x^2 + 2x + 1 = 0$$

Let $(x-1)$,

$$f(1) = -(1)^3 - 2(1)^2 + 2(1) + 1$$
$$= 0$$

$$\therefore (x-1) \text{ is a factor of } f(x)$$

Let a be an arbitrary constant

$$f(x) = (x-1)\left(-x^2 + ax - 1\right)$$

Comparing coefficient of x,

$$2 = -a - 1$$
$$a = -3$$

$$\therefore f(x) = (x-1)\left(-x^2 - 3x - 1\right)$$

For the quadratic factor,

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-1)(-1)}}{2(-1)}$$
$$= -\frac{3 \pm \sqrt{5}}{2}$$

$$\therefore x = \mathbf{1} \quad \text{or} \quad x = \mathbf{-\frac{3 \pm \sqrt{5}}{2}}$$

5. (a)

$$4x^2 + 7x - 2 = (4x - 1)(x + 2)$$

Since $4x^2 + 7x - 2$ is a factor of $f(x)$

$$f\left(\frac{1}{4}\right) = a\left[\left(\frac{1}{4}\right)^4 + 1\right] + 7\left(\frac{1}{4}\right)^3 - 10\left(\frac{1}{4}\right)^2 + b\left(\frac{1}{4}\right) = 0$$

$$\frac{257}{256}a + \frac{1}{4}b = \frac{33}{64}$$
$$257a + 64b = 132 \text{ ......(1)}$$

$$f(-2) = a\left[(-2)^4 + 1\right] + 7(-2)^3 - 10(-2)^2 + b(-2) = 0$$

$$17a - 2b = 96$$
$$b = \frac{17a - 96}{2} \text{ ......(2)}$$

Substitute Equation (2) into Equation (1),

$$257a + 64\left(\frac{17a - 96}{2}\right) = 132$$
$$257a + 544a - 3072 = 132$$
$$801a = 3204$$
$$a = 4$$

Substitute $a = 4$ into Equation (2),

$$b = \frac{17(4) - 96}{2}$$
$$= -14$$

□

(b)

$$f(x) = 4x^4 + 7x^3 - 10x^2 - 14x + 4$$

When divided by $(x + 1)$,

$$f(-1) = 4(-1)^4 + 7(-1)^3 - 10(-1)^2 - 14(-1) + 4$$
$$= \mathbf{5}$$

4 Partial Fractions

4.1 Full Solutions

1. (a)

$$\begin{aligned}\frac{8x-5}{x^2(1-x)} &= \frac{A}{x}+\frac{B}{x^2}+\frac{C}{1-x}\\ 8x-5 &= Ax(1-x)+B(1-x)+Cx^2\end{aligned}$$

When $x=1$

$$\begin{aligned}8(1)-5 &= C(1)^2\\ C &= 3\end{aligned}$$

When $x=0$,

$$\begin{aligned}8(0)-5 &= B(1)\\ B &= -5\end{aligned}$$

When $x=2$,

$$\begin{aligned}8(2)-5 &= A(2)(1-2)-5(1-2)+3(2)^2\\ A &= 3\end{aligned}$$

$$\therefore \frac{8x-5}{x^2(1-x)} = \boldsymbol{\frac{3}{x}-\frac{5}{x^2}+\frac{3}{1-x}}$$

(b)

$$\begin{aligned}\int \frac{8x-5}{x^2(1-x)}\,dx &= \int \frac{3}{x}-\frac{5}{x^2}+\frac{3}{1-x}\,dx\\ &= 3\ln x+\frac{5}{x}-3\ln(1-x)+c\\ &= \boldsymbol{3\ln\left(\frac{x}{1-x}\right)+\frac{5}{x}+c}\end{aligned}$$

2. (a)

$$\begin{aligned}\frac{3x^2+10x}{(x+2)(x^2-4)} &= \frac{3x^2+10x}{(x+2)^2(x-2)} \\ &= \frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{x-2} \\ 3x^2+10x &= A(x+2)(x-2)+B(x-2)+C(x+2)^2\end{aligned}$$

Let $x=-2$,

$$\begin{aligned}3(-2)^2+10(-2) &= B(-2-2) \\ B &= 2\end{aligned}$$

Let $x=2$,

$$\begin{aligned}3(2)^2+10(2) &= C(2+2)^2 \\ C &= 2\end{aligned}$$

Let $x=0$,

$$\begin{aligned}3(0)^2+10(0) &= A(0+2)(0-2)+2(0-2)+2(0+2)^2 \\ A &= 1\end{aligned}$$

$$\frac{3x^2+10x}{(x+2)(x^2-4)} = \boldsymbol{\frac{1}{x+2}+\frac{2}{(x+2)^2}+\frac{2}{x-2}}$$

(b) (i)

$$\begin{aligned}\int \frac{3x^2+10x}{(x+2)(x^2-4)}\,dx &= \int \frac{1}{x+2}+\frac{2}{(x+2)^2}+\frac{2}{x-2}\,dx \\ &= \boldsymbol{\ln(x+2)-\frac{2}{x+2}+2\ln(x-2)+c}\end{aligned}$$

(ii)

$$\begin{aligned}\int_3^4 \frac{3x^2+10x}{(x+2)(x^2-4)}\,dx &= \left[\ln(x+2)-\frac{2}{x+2}+2\ln(x-2)\right]_3^4 \\ &= \left[\ln 6-\frac{1}{3}+2\ln 2\right]-\left[\ln 5-\frac{2}{5}\right] \\ &= \ln\left(\frac{24}{5}\right)+\frac{1}{15}\ \textbf{(shown)}\end{aligned}$$

□

3. (a)

$$\begin{aligned}\frac{11-7x}{3x^2+11x-4} &= \frac{11-7x}{(3x-1)(x+4)}\\ &= \frac{A}{3x-1}+\frac{B}{x+4}\\ 11-7x &= A(x+4)+B(3x-1)\end{aligned}$$

Let $x=-4$,

$$\begin{aligned}11-7(-4) &= B[3(-4)-1]\\ B &= -3\end{aligned}$$

Let $x=\frac{1}{3}$,

$$\begin{aligned}11-7\left(\frac{1}{3}\right) &= A\left[\left(\frac{1}{3}\right)+4\right]\\ A &= 2\end{aligned}$$

$$\frac{11-7x}{3x^2+11x-4} = \mathbf{\frac{2}{3x-1} - \frac{3}{x+4}}$$

(b)

$$\begin{aligned}\int_1^2 \frac{11-7x}{9x^2+33x-12}\,dx &= \frac{1}{3}\int_1^2 \frac{11-7x}{3x^2+11x-4}\,dx\\ &= \frac{1}{3}\int_1^2 \frac{2}{3x-1}-\frac{3}{x+4}\,dx\\ &= \frac{1}{3}\left[\frac{2}{3}\ln(3x-1)-3\ln(x+4)\right]_1^2\\ &= \frac{1}{3}\left\{\left[\frac{2}{3}\ln 5-3\ln 6\right]-\left[\frac{2}{3}\ln 2-3\ln 5\right]\right\}\\ &= \mathbf{0.0213\ (3.s.f.)}\end{aligned}$$

4. (a) (i) Let $x = \frac{1}{3}$,

$$f\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3 + 11\left(\frac{1}{3}\right)^2 + 8\left(\frac{1}{3}\right) - 4$$
$$= 0$$

$$\therefore (3x-1) \text{ is a factor of } f(x)$$

(ii) Let a be an arbitrary constant,

$$f(x) = (3x-1)\left(x^2 + ax + 4\right)$$

Comparing coefficient of x,

$$12 - a = 8$$
$$a = 4$$

$$f(x) = (3x-1)\left(x^2 + 4x + 4\right)$$
$$= \mathbf{(3x-1)(x+2)^2}$$

(b)

$$\frac{5x^2 - 2x + 11}{3x^3 + 11x^2 + 8x - 4} = \frac{5x^2 - 2x + 11}{(3x-1)(x+2)^2}$$
$$= \frac{A}{3x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$
$$5x^2 - 2x + 11 = A(x+2)^2 + B(x+2)(3x-1) + C(3x-1)$$

Let $x = -2$,

$$5(-2)^2 - 2(-2) + 11 = C\left[3(-2) - 1\right]$$
$$C = -5$$

Let $x = \frac{1}{3}$,

$$5\left(\frac{1}{3}\right)^2 - 2\left(\frac{1}{3}\right) + 11 = A\left[\left(\frac{1}{3}\right) + 2\right]^2$$
$$A = 2$$

Let $x = 0$,

$$5(0)^2 - 2(0) + 11 = 2(0+2)^2 + B(0+2)(3(0)-1) - 5(3(0)-1)$$
$$B = 1$$

$$\therefore \frac{5x^2 - 2x + 11}{3x^3 + 11x^2 + 8x - 4} = \mathbf{\frac{2}{3x-1} + \frac{1}{x+2} - \frac{5}{(x+2)^2}}$$

(c)

$$\int \frac{5x^2 - 2x + 11}{3x^3 + 11x^2 + 8x - 4}\ dx = \int \frac{2}{3x-1} + \frac{1}{x+2} - \frac{5}{(x+2)^2}\ dx$$
$$= \mathbf{\frac{2}{3}\ln(3x-1) + \ln(x+2) + \frac{5}{x+2} + c}$$

5 Binomial Theorem

5.1 Full Solutions

1.

$$\begin{aligned}(1-2p)^n &= 1^n + \binom{n}{1}(1)^{n-1}(-2p)^1 + \binom{n}{2}(1)^{n-2}(-2p)^2 + ... \\ &= 1 - 2np + 2n(n-1)p^2 + ...\end{aligned}$$

Since the sum of the constant term, coefficient of p and p^2 is 161,

$$\begin{aligned}1 - 2n + 2n(n-1) &= 161 \\ 2n^2 - 4n - 160 &= 0 \\ n^2 - 2n - 80 &= 0 \\ (n-10)(n+8) &= 0\end{aligned}$$

$$n = \mathbf{10} \qquad \text{or} \qquad n = -8 \text{ (rej)}$$

2. (a) (i)

$$\begin{aligned}(1-x)^8 &= 1^8 + \binom{8}{1}(1)^7(-x)^1 + \binom{8}{2}(1)^6(-x)^2 + \binom{8}{3}(1)^5(-x)^3 + ... \\ &= \mathbf{1 - 8x + 28x^2 - 56x^3 + ...}\end{aligned}$$

(ii) Let $x = 2z - z^2$,

$$\begin{aligned}\left(1 - 2z - z^2\right)^8 &= 1 - 8\left(2z - z^2\right) + 28\left(2z - z^2\right)^2 - 56\left(2z - z^2\right)^3 + ... \\ &= ... + 28\left(4z^2 - 4z^3 + ...\right) - 56\left(8z^3 + ...\right) + ... \\ &= ... - 28\left(4z^3\right) - 56\left(8z^3\right) + ... \\ &= ... - 560z^3 + ...\end{aligned}$$

Hence, the coefficient of z^3 is $\mathbf{-560}$

(b) (i)

$$\begin{aligned}\left(2x - \frac{1}{3x^3}\right)^6 &= \binom{6}{r}(2x)^{6-r}\left(-\frac{1}{3x^3}\right)^r \\ &= \binom{6}{r}(2)^{6-r}(x)^{6-r}\left(-\frac{1}{3}\right)^r(x)^{-3r} \\ &= \boldsymbol{\binom{6}{r}(2)^{6-r}\left(-\frac{1}{3}\right)^r x^{6-4r}}\end{aligned}$$

(ii) For the constant term, powers of x is 0

$$\begin{aligned}\therefore 6 - 4r &= 0 \\ r &= \frac{3}{2} \notin \mathbb{Z}^+\end{aligned}$$

Since the value of r is not a positive integer, there is **no constant term**

(iii) Comparing terms,

For the x^4 term, we need the $\dfrac{1}{x^2}$ term

$$\therefore x^{-2} = x^{6-4r}$$
$$r = 2$$

For the constant term, we need the x^2 term

$$\therefore x^{2} = x^{6-4r}$$
$$r = 1$$

For the $\dfrac{3}{x}$ term, we need the x^3 term

$$\therefore x^{3} = x^{6-4r}$$
$$r = \frac{3}{4} \notin \mathbb{Z}^+$$

There is no x^3 term

$$\begin{aligned}
&\left(3x^4 + 2 - \frac{3}{x}\right)\left(2x - \frac{1}{3x^3}\right)^6 \\
&= \left(3x^4 + 2 - \frac{3}{x}\right)\left[\binom{6}{1}(2)^{6-1}\left(-\frac{1}{3}\right)^r x^2 + \binom{6}{2}(2)^{6-2}\left(-\frac{1}{3}\right)^r x^{-2} + ...\right] \\
&= \left(3x^4 + 2 - \frac{3}{x}\right)\left[-64x^2 + \frac{80}{3x^2} + ...\right] \\
&= ... + 2\left(-64x^2\right) + \left(3x^4\right)\left(\frac{80}{3x^2}\right) + ... \\
&= ... - 48x^2 + ...
\end{aligned}$$

Hence, the coefficient of x^2 is $\mathbf{-48}$

3. (a)

$$
\begin{aligned}
T_{r+1} &= \binom{10}{r}\left(x^2\right)^{10-r}\left(-\frac{1}{2x^3}\right)^r \\
&= \binom{10}{r}\left(-\frac{1}{2}\right)^{10}\left(x^{20-5r}\right)
\end{aligned}
$$

For the independent term, x^0

$$
\begin{aligned}
20 - 5r &= 0 \\
r &= 4
\end{aligned}
$$

$$
\begin{aligned}
\therefore \text{Term independent of } x &= \binom{10}{4}\left(-\frac{1}{2}\right)^{10} \\
&= \mathbf{13\frac{1}{8}}
\end{aligned}
$$

(b) (i)

$$
\begin{aligned}
\left(2x+\frac{1}{4}\right)^9 &= ... + \binom{9}{2}(2x)^{9-2}\left(\frac{1}{4}\right)^2 + \binom{9}{3}(2x)^{9-3}\left(\frac{1}{4}\right)^3 + ... \\
&= ... + 288x^7 + 84x^6 + ...
\end{aligned}
$$

$$a = \mathbf{288} \qquad b = \mathbf{84}$$

(ii)

$$
\begin{aligned}
\left(2x+\frac{1}{4}\right)^9\left(\frac{4}{x}-1\right)\left(\frac{4}{x}+1\right) &= \left(2x+\frac{1}{4}\right)^9\left(\frac{16}{x^2}-1\right) \\
&= \left(... + 576x^8 + ... + 84x^6 + ...\right)\left(\frac{16}{x^2}-1\right) \\
&= ... + \left[(576)(16) - 84\right]x^6 + ... \\
&= ... + 9132x^6 + ...
\end{aligned}
$$

$$\therefore \text{Coefficient of } x^6 = \mathbf{9132}$$

4. (a) (i)

$$\begin{aligned}(1+6x)^6 &= 1^6 + \binom{6}{1}(1)^{6-1}(6x)^1 + \binom{6}{2}(1)^{6-2}(6x)^2 + ...\\ &= \mathbf{1 + 36x + 540x^2 + ...}\end{aligned}$$

(ii)

$$\begin{aligned}(1-kx)^6 &= 1^6 + \binom{6}{1}(1)^{6-1}(-kx)^1 + \binom{6}{2}(1)^{6-2}(-kx)^2 + ...\\ &= \mathbf{1 - 6kx + 15k^2x^2 + ...}\end{aligned}$$

(b) By observation,

$$1 + (6-k)x - 6kx^2 = (1+6x)(1-kx)$$

$$\begin{aligned}\therefore \left[1 + (6-k)x - 6kx^2\right]^6 &= (1+6x)^6(1-kx)^6\\ &= \left(1 + 36x + 540x^2 + ...\right)\left(1 - 6kx + 15k^2x^2 + ...\right)\\ &= ... + \left[1\left(15k^2\right) + 36(-6k) + 540\right]x^2 + ...\end{aligned}$$

$$\therefore \text{Coefficient of } x^2 = \left(\mathbf{15k^2 - 216k + 540}\right)$$

(c) Hence,

$$\begin{aligned}15k^2 - 216k + 540 &= 168\\ 15k^2 - 216k + 372 &= 0\\ (3k-6)(5k-62) &= 0\end{aligned}$$

$$\therefore k = \mathbf{2} \qquad \text{or} \qquad k = 12\frac{2}{5} \text{ (rej)}$$

5. (a)

$$T_{r+1} = \boldsymbol{\binom{n}{r}\left(\frac{x}{2}\right)^{n-r}\left(-\frac{k}{x^2}\right)^{r}}$$

(b)

$$\begin{aligned} T_{r+1} &= \binom{n}{r}\left(\frac{x}{2}\right)^{n-r}\left(-\frac{k}{x^2}\right)^{r} \\ &= \binom{n}{r}\left(\frac{1}{2}\right)^{n-r}(-k)^r(x)^{n-3r} \end{aligned}$$

For the independent term, x^0,

$$\begin{aligned} n - 3r &= 0 \\ n &= 3r \end{aligned}$$

Since r is a positive integer, $3r$ is a multiple of 3, n is a multiple of 3

□

(c) Since $n = 9$,

$$r = 3$$

$$\begin{aligned} \binom{9}{3}\left(\frac{1}{2}\right)^{9-3}(-k)^3 &= -\frac{2625}{2} \\ k^3 &= 1000 \\ k &= \mathbf{10} \end{aligned}$$

(d) To get the term independent of x, we need to find the x^0 and $\frac{1}{x^3}$ for $\left(\frac{x}{2} - \frac{k}{x^2}\right)^9$

For $\frac{1}{x^3}$ term,

$$\begin{aligned} 9 - 3r &= -3 \\ r &= 4 \end{aligned}$$

$$\begin{aligned} (2 + x^3)\left(\frac{x}{2} - \frac{k}{x^2}\right)^9 &= (2 + x^3)\left[... - \frac{2625}{2} + \binom{9}{4}\left(\frac{1}{2}\right)^{9-4}(-10)^4x^{-3} + ...\right] \\ &= (2 + x^3)\left(... - \frac{2625}{2} + \frac{39375}{x^3} + ...\right) \\ &= ... + \left[2\left(-\frac{2625}{2}\right) + 39375\right] + ... \\ &= ... + 36750 + ... \end{aligned}$$

$\therefore$ Term independent of $x = \mathbf{36750}$

6 Exponential & Logarithms

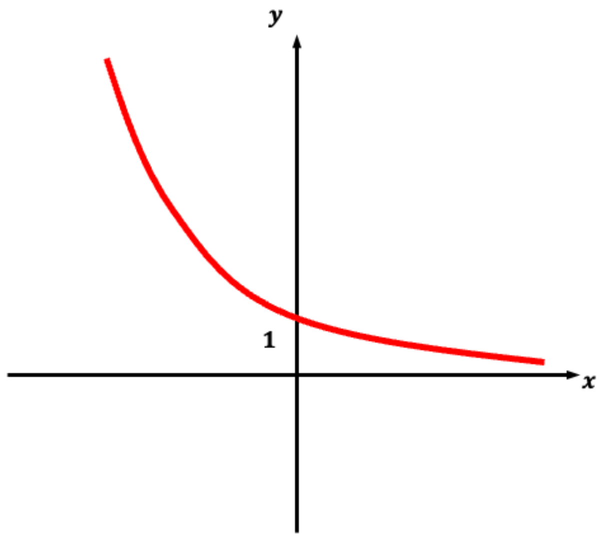
6.1 Full Solutions

1. (a)

$$\begin{aligned}\log_3 4 - \log_9\left(x^2+4x+4\right) &= \log_{\frac{1}{3}} x\\ \log_3 4 - \frac{2\log_3(x+2)}{\log_3 9} &= \frac{\log_3 x}{\log_3 \frac{1}{3}}\\ \log_3 4 - \log_3(x+2) &= -\log_3 x\end{aligned}$$

$$\begin{aligned}\therefore \frac{4}{x+2} &= \frac{1}{x}\\ x &= \mathbf{\frac{2}{3}}\end{aligned}$$

(b) (i) Graph



(ii)

$$\begin{aligned}\ln\left(\frac{1}{\sqrt{x-3}}\right) &= \frac{1}{2}x\\ \ln 1 - \frac{1}{2}\ln(x-3) &= \frac{1}{2}x\\ \ln(x-3) &= -x\\ x-3 &= e^{-x}\end{aligned}$$

$$\therefore \boldsymbol{y = x - 3}$$

2. (a)

$$\lg(by)^a = \lg\left(\frac{125}{y}\right) + \lg y^4$$
$$= \lg\left(125y^3\right)$$
$$= 3\lg(5y)$$

$$a = \mathbf{3} \qquad b = \mathbf{5}$$

(b)

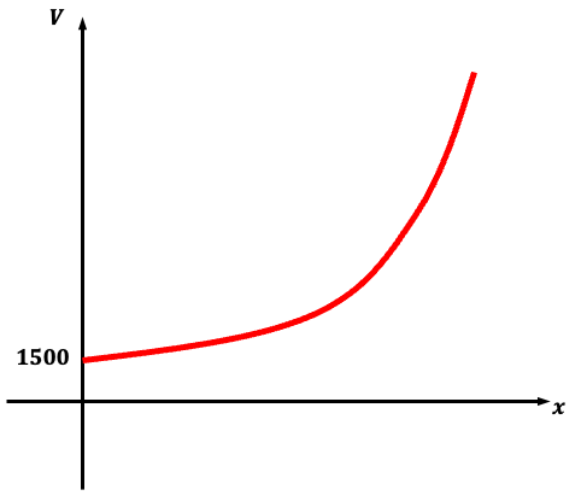
$$2\log_5 e^x + \frac{1}{\log_2 5} = \log_5\left(2 - 3e^x\right)$$
$$\log_5 e^{2x} + \log_5 2 = \log_5\left(2 - 3e^x\right)$$
$$\log_5 2e^{2x} = \log_5\left(2 - 3e^x\right)$$

$$\therefore 2e^{2x} + 3e^x - 2 = 0$$
$$\left(2e^x - 1\right)\left(e^x + 2\right) = 0$$
$$\therefore e^x = \frac{1}{2} \qquad \text{or} \qquad e^x = -2 \text{ (N.A.)}$$
$$\therefore x = \mathbf{\ln\frac{1}{2}}$$

3. (a)

$$\boldsymbol{V = 1500(2)^{\frac{x}{5}}}$$

(b) Graph



(c) For the value to reach $1 000 000

$$1500(2)^{\frac{x}{5}} = 1000000$$
$$2^{\frac{x}{5}} = 666\frac{2}{3}$$
$$\therefore x = 5\left[\frac{\ln\left(666\frac{2}{3}\right)}{\ln 2}\right]$$
$$= 46.904108...$$
$$= \mathbf{46.9 \text{ years}}$$

4. (a)

$$\log_a 2y^2 + \log_a 8 + \log_a 16y - \log_a 64y = 2\log_a 4$$
$$\log_a 2 + \log_a y^2 + \log_a 2^3 + \log_a 2^4 + \log_a y - \log_a 2^6 - \log_a y = 2\log_a 2^2$$

$$2\log_a 2 + 2\log_a y = 4\log_a 2$$
$$2\log_a y = 2\log_a 2$$

Comparing terms,

$$y = \mathbf{2}$$

(b)

$$x = \lg m \quad \Rightarrow \quad m = 10^x$$

$$10^{2x+1} + 7\left(10^x\right) = 26$$
$$10\left(10^{2x}\right) + 7\left(10^x\right) = 26$$

By substitution,

$$10m^2 + 7m - 26 = 0$$
$$(10m - 13)(m + 2) = 0$$

$$\therefore m = \mathbf{1\frac{3}{10}} \qquad \text{or} \qquad m = -2 \text{ (N.A.)}$$

5. (a) Initially, $t = 0$,

$$N = \frac{1000}{1 + 199e^{-0.8(0)}}$$
$$= \mathbf{5}$$

(b) When $N = 937$, $t = x$

$$937 = \frac{1000}{1 + 199e^{-0.8x}}$$
$$e^{-0.8x} = -\frac{\left(1 - \frac{1000}{937}\right)}{199}$$
$$= \frac{63}{186463}$$

$$\therefore x = \frac{\left[\ln\left(\frac{63}{186463}\right)\right]}{-0.8}$$
$$= 9.991066...$$
$$= \mathbf{10\ (nearest\ whole\ number)}$$

(c) As $t \to \infty$, $e^{-0.8t} \to 0$

$$\therefore \lim_{t\to\infty}\left(\frac{1000}{1 + 199e^{-0.8t}}\right) = \frac{1000}{1 + 0}$$
$$= \mathbf{1000}$$

7 Trigonometry

7.1 Full Solutions

1. (a) (i)

$$\begin{aligned}\sin^2 y + 2\cos 2y &= 2\cos y\\ 1 - \cos^2 y + 4\cos^2 y - 2 - 2\cos y &= 0\\ 3\cos^2 y - 2\cos y - 1 &= 0\\ (3\cos y + 1)(\cos y - 1) &= 0\end{aligned}$$

$$\cos y = -\frac{1}{3} \qquad \text{or} \qquad \cos y = 1$$

For $\cos y = -\dfrac{1}{3}$,

$$\text{Basic angle } \alpha = \cos^{-1}\left(\frac{1}{3}\right) \quad \text{(Quadrant 2 and 3)}$$

$$\begin{aligned}x &= 180^\circ - \cos^{-1}\left(\frac{1}{3}\right)\\ &= 109.471220...\\ &= \mathbf{109.5^\circ\ (1.d.p.)}\end{aligned}$$

$$\begin{aligned}x &= 180^\circ + \cos^{-1}\left(\frac{1}{3}\right)\\ &= 250.528779...\\ &= \mathbf{250.5^\circ\ (1.d.p.)}\end{aligned}$$

For $\cos y = 1$,

$$\begin{aligned}\text{Basic angle } \alpha &= \cos^{-1}(1)\\ &= 0 \quad \text{(Quadrant 1 and 4)}\end{aligned}$$

$$x = \mathbf{0^\circ} \qquad x = \mathbf{360^\circ}$$

(ii)

$$\begin{aligned}\text{LHS} &= \frac{\cos(A+B) + \cos(A-B)}{\sin(A+B) - \sin(A-B)}\\ &= \frac{\cos A\cos B + \sin A\sin B + \cos A\cos B - \sin A\sin B}{\sin A\cos B + \cos A\sin B - \sin A\cos B + \cos A\sin B}\\ &= \frac{2\cos A\cos B}{2\cos A\sin B}\\ &= \frac{\cos B}{\sin B}\\ &= \cot B\\ &= \text{RHS } \textbf{(shown)}\end{aligned}$$

□

(b)

$$\begin{aligned}\tan\left(\frac{\pi}{6}\right) &= \frac{4}{MB}\\ AM = MB &= \frac{4}{\tan\left(\frac{\pi}{6}\right)}\\ &= 4\sqrt{3}\end{aligned}$$

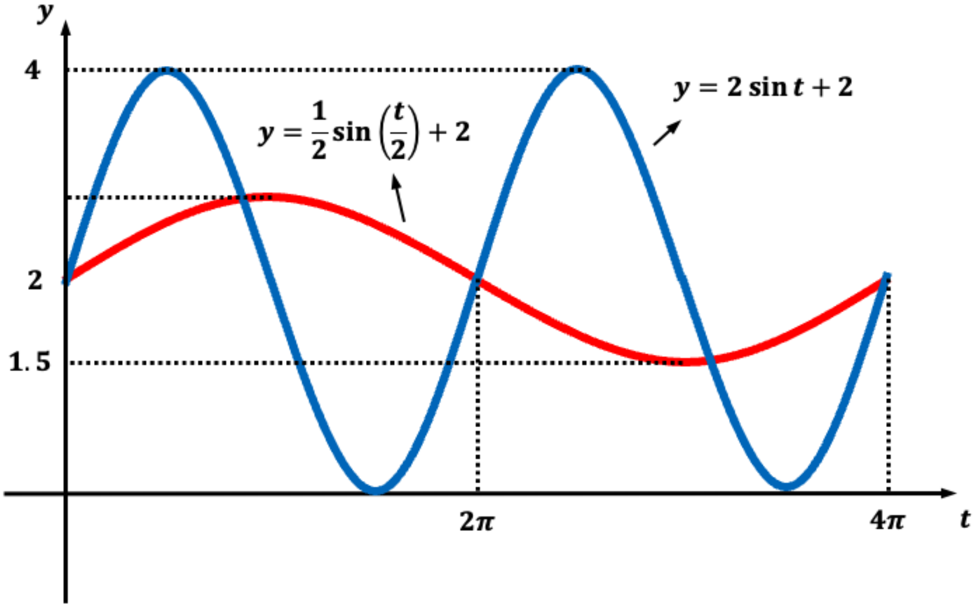
$$\begin{aligned}AC &= \sqrt{\left(4\sqrt{3}+4\sqrt{3}\right)^2+(4)^2}\\ &= 4\sqrt{13}\end{aligned}$$

By sine rule,

$$\begin{aligned}\frac{\sin\angle ACM}{4\sqrt{3}} &= \frac{\sin\left(\frac{5\pi}{6}\right)}{4\sqrt{13}}\\ \therefore \sin\angle ACM &= \frac{2}{4\sqrt{13}}\times 4\sqrt{3}\\ &= \frac{\sqrt{3}}{2\sqrt{13}}\times\frac{\sqrt{13}}{\sqrt{13}}\\ &= \frac{\sqrt{39}}{26}\end{aligned}$$

$$\begin{aligned}\angle ACM &= \sin^{-1}\left(\frac{\sqrt{39}}{26}\right)\\ \therefore k &= \mathbf{39}\end{aligned}$$

2. (a) Graph



(b) (i) In the interval $0 < t < 4\pi$, the two graphs have **3 points of intersection**, showing that there were **3 instances** when the waves on the two days reached the same height

(ii) **Day 1** would have provided surfers with a more thrilling experience of riding the waves at sea. **Period of the waves in Day 1 is half of the period of the waves in Day 2**, showing that for Day 1 there is an additional cycle between the greatest height and the least height of the waves within the same time interval.

In addition, **the amplitude of the waves in Day 1 is greater than the amplitude of the waves in Day 2**, showing that the rise and drop in height of the waves in Day 1 is greater

3. (a)

$$\begin{aligned}\text{LHS} &= \frac{2\cos 2A + \cos A + 2}{2\sin 2A + \sin A} \\ &= \frac{2\left(2\cos^2 A - 1\right) + \cos A + 2}{2\left(2\sin A\cos A\right) + \sin A} \\ &= \frac{4\cos^2 A + \cos A}{\sin A\left(4\cos A + 1\right)} \\ &= \frac{\cos A\left(4\cos A + 1\right)}{\sin A\left(4\cos A + 1\right)} \\ &= \frac{\cos A}{\sin A} \\ &= \cot A \\ &= \text{RHS } \textbf{(shown)}\end{aligned}$$

□

(b) From part (a),

$$A = 3x$$

$$\therefore \cot 3x = 5 \quad \Rightarrow \quad \tan 3x = \frac{1}{5}$$

$$\text{Basic angle } \alpha = \tan^{-1}\left(\frac{1}{5}\right) \qquad \text{(Quadrant 1 and 3)}$$

$$\begin{aligned}3x &= \tan^{-1}\left(\frac{1}{5}\right) \\ x &= \frac{\tan^{-1}\left(\frac{1}{5}\right)}{3} \\ &= 0.0657985... \\ &= \mathbf{0.0658\ (3.s.f.)}\end{aligned}$$

$$\begin{aligned}3x &= \pi + \tan^{-1}\left(\frac{1}{5}\right) \\ x &= \frac{\pi + \tan^{-1}\left(\frac{1}{5}\right)}{3} \\ &= 1.11299... \\ &= \mathbf{1.11\ (3.s.f.)}\end{aligned}$$

$$\begin{aligned}3x &= 2\pi + \tan^{-1}\left(\frac{1}{5}\right) \\ x &= \frac{2\pi + \tan^{-1}\left(\frac{1}{5}\right)}{3} \\ &= 2.160193... \\ &= \mathbf{2.16\ (3.s.f.)}\end{aligned}$$

4. (a) (i)

$$\begin{aligned} \text{LHS} &= \frac{(\cos\theta + \sin\theta)^2}{\sec^2\theta + 2\tan\theta} \\ &= \frac{[\cos\theta\,(1+\tan\theta)]^2}{(1+\tan\theta)^2} \\ &= \cos^2\theta \\ &= \text{RHS } \textbf{(shown)} \end{aligned}$$

□

(ii)

$$\frac{1}{\cos^2\theta} = 2\tan\theta + 4$$

$$\begin{aligned} \sec^2\theta - 2\tan\theta - 4 &= 0 \\ \tan^2\theta - 2\tan\theta - 3 &= 0 \\ (\tan\theta - 3)(\tan\theta + 1) &= 0 \end{aligned}$$

$$\tan\theta = 3 \qquad \text{or} \qquad \tan\theta = -1$$

For $\tan\theta = 3$,

$$\text{Basic angle } \alpha = \tan^{-1}(3) \quad \text{(Quadrant 1 and 3)}$$

$$\begin{aligned} \theta &= \tan^{-1}(3) \\ &= 1.24904... \\ &= \mathbf{1.25\ (3.s.f.)} \end{aligned}$$

$$\begin{aligned} \theta &= \pi + \tan^{-1}(3) \\ &= 4.390638... \\ &= \mathbf{4.39\ (3.s.f.)} \end{aligned}$$

For $\tan\theta = 3$,

$$\begin{aligned} \text{Basic angle } \alpha &= \tan^{-1}(1) \\ &= \frac{\pi}{4} \quad \text{(Quadrant 2 and 4)} \end{aligned}$$

$$\begin{aligned} \theta &= \pi - \frac{\pi}{4} \\ &= \boldsymbol{\frac{3\pi}{4}} \end{aligned}$$

$$\begin{aligned} \theta &= 2\pi - \frac{\pi}{4} \\ &= \boldsymbol{\frac{7\pi}{4}} \end{aligned}$$

(b) (i)

$$PQ = 5\sin\theta \qquad QR = 5\cos\theta$$

$$\begin{aligned}
\text{Area of shaded region } A &= \frac{1}{2}(5\sin\theta)(5\cos\theta) - \frac{25}{4}\cos^2\theta \\
&= \frac{25}{2}\sin\theta\cos\theta - \frac{25}{4}\cos^2\theta \\
&= \frac{25}{4}\sin 2\theta - \frac{25}{4}\left(\frac{\cos 2\theta + 1}{2}\right) \\
&= \frac{50\sin 2\theta - 25\cos 2\theta - 25}{8} \\
&= \frac{25}{8}(2\sin 2\theta - \cos 2\theta) - \frac{25}{8} \ \textbf{(shown)}
\end{aligned}$$

□

(ii)

$$\begin{aligned}
r &= \sqrt{(2)^2 + (1)^2} \\
&= \sqrt{5}
\end{aligned}$$

$$\begin{aligned}
\alpha &= \tan^{-1}\left(\frac{1}{2}\right) \\
&= 26.5651...^\circ
\end{aligned}$$

$$\therefore A = \boldsymbol{\frac{25\sqrt{5}}{8}\sin(2\theta - 26.6^\circ) - \frac{25}{8}}$$

(iii)

$$A_{\max} = \boldsymbol{\left(\frac{25\sqrt{5} - 25}{8}\right)}$$

5. (a)

$$\begin{aligned}\text{LHS} &= \frac{1-\tan^2\theta}{1+\tan^2\theta}\\ &= \frac{1-\left(\sec^2\theta-1\right)}{\sec^2\theta}\\ &= \frac{2-\sec^2\theta}{\sec^2\theta}\\ &= 2\cos^2\theta-1\\ &= \cos 2\theta\\ &= \text{RHS } \textbf{(shown)}\end{aligned}$$

□

(b) By manipulation,

$$\begin{aligned}1+x^2 &= \sqrt{2}x^2-\sqrt{2}\\ \frac{1-x^2}{1+x^2} &= -\frac{1}{\sqrt{2}}\end{aligned}$$

By comparing with part (a),

$$x=\tan\theta \qquad \theta = 67.5^\circ$$

$$\begin{aligned}\therefore \cos 2(67.5^\circ) &= \cos 135^\circ\\ &= \cos\left(90^\circ+45^\circ\right)\\ &= -\frac{1}{\sqrt{2}} \textbf{ (shown)}\end{aligned}$$

□

(c)

$$\begin{aligned}1+x^2 &= \sqrt{2}x^2-\sqrt{2}\\ x^2-\sqrt{2}x^2 &= -1-\sqrt{2}\\ x^2\left(1-\sqrt{2}\right) &= -1-\sqrt{2}\\ x^2 &= \frac{1+\sqrt{2}}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}\\ &= (1+\sqrt{2})^2\end{aligned}$$

$$\therefore x = \tan 67.5^\circ = 1+\sqrt{2}$$

$$c=d=\mathbf{1}$$

(d)

$$\begin{aligned}\tan 7.5^\circ &= \tan(67.5^\circ-60^\circ)\\ &= \frac{\tan 67.5^\circ-\tan 60^\circ}{1+(\tan 67.5^\circ)(\tan 60^\circ)}\\ &= \frac{\left(1+\sqrt{2}\right)-\sqrt{3}}{1+\left(1+\sqrt{2}\right)\left(\sqrt{3}\right)}\\ &= \frac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{3}+\sqrt{6}} \textbf{ (shown)}\end{aligned}$$

□

8 Coordinate Geometry

8.1 Full Solutions

1. (a)

$$\begin{aligned}\text{Length of } AC &= \text{Length of } CD \\ &= \sqrt{(15-0)^2 + (0-8)^2} \\ &= 17 \text{ units}\end{aligned}$$

$$A = (15-17, 0) = (-2, 0) \ \textbf{(shown)}$$

□

(b)

$$\begin{aligned}\text{Gradient of } BC &= \frac{8-0}{0-15} \\ &= -\frac{8}{15}\end{aligned}$$

$$\therefore \text{Gradient of perpendicular line of } BC = \frac{15}{8}$$

$$\therefore y - 0 = \frac{15}{8}(x+2)$$

$$\boldsymbol{y = \frac{15}{8}x + \frac{15}{4}}$$

(c) Equation of line BC:

$$\begin{aligned}y - 8 &= -\frac{8}{15}(x+0) \\ y &= -\frac{8}{15}x + 8\end{aligned}$$

Let $B(-2, y)$,

$$\begin{aligned}y &= -\frac{8}{15}(-2) + 8 \\ &= 9\frac{1}{15}\end{aligned}$$

$$\therefore B\left(-2, 9\frac{1}{15}\right)$$

ABC, ACE share the same base AC. Hence, the perpendicular height of E to x-axis is $\frac{1}{2}AB$

$$\begin{aligned}\text{Perpendicular height of } E &= \frac{1}{2}\left(9\frac{1}{15}\right) \\ &= 4\frac{8}{15}\end{aligned}$$

Let $E\left(x, -4\frac{8}{15}\right)$

$$\begin{aligned}-4\frac{8}{15} &= -\frac{8}{15}(x) + 8 \\ x &= 23\frac{1}{2}\end{aligned}$$

$$\therefore \boldsymbol{E\left(23\frac{1}{2}, -4\frac{8}{15}\right)}$$

(d)

$$\begin{aligned}
\text{Area of } ABFE &= 2\,(\text{Area of } ABE) \\
&= 2\left[\frac{1}{2}\begin{vmatrix} -2 & -2 & \dfrac{47}{2} & -2 \\ 0 & \dfrac{136}{15} & -\dfrac{68}{15} & 0 \end{vmatrix}\right] \\
&= \left|-\frac{136}{15} - \frac{3332}{15}\right| \\
&= \mathbf{231\frac{1}{5}\ units^2}
\end{aligned}$$

2. (a)

$$\begin{aligned}
\text{Gradient of } BC &= \text{Gradient of } AD \\
&= \frac{5-2}{-1-3} \\
&= -\frac{3}{4}
\end{aligned}$$

Equation of BC:

$$\begin{aligned}
y - 3 &= -\frac{3}{4}(x-5) \\
y &= -\frac{3}{4}x + 6\frac{3}{4} \text{ ......(1)}
\end{aligned}$$

$$\begin{aligned}
\text{Gradient of } CD &= \frac{3-2}{5-3} \\
&= \frac{1}{2}
\end{aligned}$$

Equation of CD:

$$\begin{aligned}
y - 5 &= \frac{1}{2}(x+1) \\
y &= -2x + 3 \text{ ......(2)}
\end{aligned}$$

Let Equation (1) = Equation (2),

$$\begin{aligned}
\frac{3}{4}x + 6\frac{3}{4} &= -2x + 3 \\
5x &= -15 \\
x &= -3
\end{aligned}$$

Substitute $x = -3$ into Equation (2),

$$\begin{aligned}
y &= -2(-3) + 3 \\
&= 9
\end{aligned}$$

$$\therefore \boldsymbol{B(-3, 9)}$$

(b)

$$\begin{aligned}\text{Area of trapezium } ABCD &= \frac{1}{2}\begin{vmatrix}5 & -3 & -1 & 3 & 5\\ 3 & 9 & 5 & 2 & 3\end{vmatrix}\\ &= \frac{1}{2}\left|37-7\right|\\ &= \mathbf{15\ units^2}\end{aligned}$$

(c) We shall first prove that $\triangle AED$ and $\triangle BEC$ are similar

$$\angle BED \text{ is a common angle (A)}$$

$$\angle ADE = \angle BCE \text{ (corresponding angles) (A)}$$

$$\therefore \text{By the AA similarity test, } \triangle AED \text{ and } \triangle BEC \text{ are similar}$$

□

Since $\triangle AED$ and $\triangle BEC$ are similar,

$$\begin{aligned}\frac{ED}{EC} &= \frac{EA}{EB}\\ &= \sqrt{\frac{1}{4}}\\ &= \frac{1}{2}\end{aligned}$$

Hence, by the Midpoint theorem, A and D are the midpoints of EB and EC respectively

Let $E(m, n)$,

$$\left(\frac{5+m}{2}, \frac{3+n}{2}\right) = (3, 2)$$

$$m = n = 1$$

$$\therefore \boldsymbol{E(1, 1)}$$

3. (a) Since U is on the x-axis, let U be $(a, 0)$

$$\begin{aligned}\frac{4-0}{-8-a} &= \frac{0-(-2)}{a-10} \\ 4a-40 &= -16-2a \\ 6a &= 24 \\ a &= 4\end{aligned}$$

$$\therefore \boldsymbol{U(4,0)}$$

(b) Let the coordinates of P be (b, c)

$$\begin{aligned}\frac{c-4}{b-(-8)} &= -\frac{1}{\left(\frac{c-9}{b-(-3)}\right)} \\ \frac{c-4}{b+8} &= \frac{b+3}{9-c} \\ 13c-c^2-36 &= b^2+11b+24 \\ b^2+c^2+11b-13c+60 &= 0 \text{ ......(1)}\end{aligned}$$

$$\begin{aligned}\frac{4-(-2)}{-8-10} &= \frac{c-9}{b-(-3)} \\ -\frac{1}{3} &= \frac{c-9}{b+3} \\ b &= 24-3c \text{ ......(2)}\end{aligned}$$

Substitute Equation (2) into Equation (1),

$$\begin{aligned}(24-3c)^2+c^2+11(24-3c)-13c+60 &= 0 \\ 576-144c+9c^2+c^2+264-33c-13c+60 &= 0 \\ 10c^2-190c+900 &= 0 \\ c^2-19c+90 &= 0 \\ (c-9)(c-10) &= 0\end{aligned}$$

$$c = 9 \text{ (rej)} \qquad \text{or} \qquad c = 10$$

Substitute $c = 10$ into Equation (2),

$$\begin{aligned}b &= 24-3(10) \\ &= -6\end{aligned}$$

$$\therefore \boldsymbol{P(-6,10)}$$

(c) Both triangles share a common height

$$\begin{aligned}\therefore \frac{\text{Area of } \triangle RSU}{\text{Area of } \triangle STU} &= \frac{|RU|}{|UT|} \\ &= \frac{\sqrt{(4-10)^2+(0+2)^2}}{\sqrt{(-8-4)^2+(4-0)^2}} \\ &= \frac{\sqrt{40}}{\sqrt{160}} \\ &= \boldsymbol{\frac{1}{2}}\end{aligned}$$

(d)

$$\begin{aligned}\text{Area of } PQRT &= \frac{1}{2}\begin{vmatrix} -6 & -3 & 10 & -8 & -6 \\ 10 & 9 & -2 & 4 & 10 \end{vmatrix} \\ &= \frac{1}{2}\left|-88-52\right| \\ &= \mathbf{70\ units^2}\end{aligned}$$

(e) By inspection,

$$W\,(7,-1)$$

$$\begin{aligned}\therefore \frac{\text{Area of parallelogram } PQRW}{\text{Area of trapezium } PQRT} &= \frac{\text{Area of trapezium } PQRT - \text{Area of } \triangle PWT}{\text{Area of trapezium } PQRT} \\ &= \frac{70 - \left[\frac{1}{2}\begin{vmatrix} -6 & 7 & -8 & -6 \\ 10 & -1 & 4 & 10 \end{vmatrix}\right]}{70} \\ &= \frac{70 - \left[\frac{1}{2}\left|-46-54\right|\right]}{70} \\ &= \mathbf{\frac{2}{7}}\end{aligned}$$

4. (a)

$$\boldsymbol{C(7,7)}$$

$$\begin{aligned}\text{Midpoint of } BD &= \left(\frac{0+7}{2}, \frac{3+0}{2}\right)\\ &= \left(\frac{7}{2}, \frac{3}{2}\right)\end{aligned}$$

$$\begin{aligned}\text{Gradient of } BD &= \frac{3-0}{0-7}\\ &= -\frac{3}{7}\end{aligned}$$

$$\therefore \text{Gradient of perpendicular bisector} = \frac{7}{3}$$

Equation of perpendicular bisector of BD:

$$\begin{aligned}y - \frac{3}{2} &= \frac{7}{3}\left(x - \frac{7}{2}\right)\\ 3y &= 7x - 20 \text{ ......(1)}\\ y &= \frac{2x-14}{5} \text{ ......(2)}\end{aligned}$$

Substitute Equation (2) into Equation (1),

$$\begin{aligned}3\left(\frac{2x-14}{5}\right) &= 7x - 20\\ 6x - 42 &= 35x - 100\\ x &= 2\end{aligned}$$

Substitute $x = 2$ into Equation (2),

$$\begin{aligned}y &= \frac{2(2)-14}{5}\\ &= -2\end{aligned}$$

$$\therefore \boldsymbol{A(2,-2)}$$

(b)

$$\begin{aligned}\text{Area of quadrilateral } ABCD &= \frac{1}{2}\begin{vmatrix} 2 & 7 & 7 & 0 & 2\\ -2 & 0 & 7 & 3 & -2\end{vmatrix}\\ &= \frac{1}{2}\left|70 - (-8)\right|\\ &= \mathbf{39\ units^2}\end{aligned}$$

(c)

$$\begin{aligned}\text{Gradient of } AC &= \frac{7+2}{7-2}\\ &= \frac{9}{5}\end{aligned}$$

Since

$$-\frac{3}{7} \times \frac{9}{5} \neq -1$$

Since AC and BD are not perpendicular to each other, $ABCD$ is not a kite

9 Further Coordinate Geometry

9.1 Full Solutions

1. (a)

$$x^2 + y^2 + 2x - 6y - 15 = 0$$
$$x^2 + y^2 + 2(1)x + 2(-3)y + (-15) = 0$$

$$\begin{aligned}\text{Radius} &= \sqrt{(1)^2 + (-3)^2 - (-15)} \\ &= \sqrt{25} \\ &= \mathbf{5\ m\ (shown)}\end{aligned}$$

□

(b)

$$\text{Centre} = (-1, 3)$$

$$\begin{aligned}\text{Length from P to the centre} &= \sqrt{(-5+1)^2 + (8-3)^2} \\ &= \sqrt{41} \\ &= 6.40312...\text{ m} > 5\text{ m}\end{aligned}$$

Since the distance between P to the centre, is larger than the radius of the lawn, the lamp lost P lies **outside** the lawn

(c) By inspection,

$$R(2, 7)$$

$$\begin{aligned}\text{Gradient of radius to } R &= \frac{7-3}{2-(-1)} \\ &= \frac{4}{3}\end{aligned}$$

$$\therefore \text{Gradient of tangent} = -\frac{3}{4}$$

Equation of tangent:

$$y - 7 = -\frac{3}{4}(x - 2)$$

$$\therefore \boldsymbol{y = -\frac{3}{4}x + \frac{17}{2}}$$

2. (a)

$$x^2 + y^2 - 2kx + 2y + 1 = 0$$
$$x^2 + y^2 + 2(-k)x + 2(1)y + 1 = 0$$

$$\text{Centre} = (k, -1)$$

$$\text{Radius} = \sqrt{(-k)^2 + (-1)^2 - 1}$$
$$2 = \sqrt{k^2}$$
$$k = \mathbf{2}$$

(b)

$$\text{Gradient of chord} = \frac{2 - (-1)}{3 - 0}$$
$$= 1$$

$$\therefore \text{Gradient of perpendicular bisector of chord} = -1$$

$$\text{Midpoint of chord} = \left(\frac{0+3}{2}, \frac{2+(-1)}{2}\right)$$
$$= \left(\frac{3}{2}, \frac{1}{2}\right)$$

Equation of perpendicular bisector:

$$y - \frac{1}{2} = -\left(x - \frac{3}{2}\right)$$
$$y = -x + 2 \text{ ......(1)}$$
$$y = 2x + 2 \text{ ......(2)}$$

Let Equation (1) = Equation (2),

$$2x + 2 = -x + 2$$
$$x = 0$$

Substitute $x = 0$ into Equation (1),

$$y = 2$$
$$\therefore \text{Centre of } C_2 = (0, 2)$$

By inspection

$$\text{Radius} = 3 \text{ units}$$
$$\therefore \boldsymbol{x^2 + (y - 2)^2 = 9}$$

3. (a)

$$x^2 + y^2 + 2x + 4y - 20 = 0$$
$$x^2 + y^2 + 2(1)x + 2(2)y + (-20) = 0$$

$$\text{Centre} = (-1, -2)$$

$$\text{Gradient of normal} = \frac{2-(-2)}{2-(-1)} = \frac{4}{3}$$

$$\therefore \text{Gradient of tangent} = -\frac{3}{4}$$

Equation of tangent:

$$y - 2 = -\frac{3}{4}(x-2)$$

$$\boldsymbol{y = -\frac{3}{4}x + \frac{7}{2}}$$

(b) (i)

$$\text{Gradient of } AB = \frac{3-2}{9-0} = \frac{1}{9}$$

$$\text{Gradient of } AC = \frac{2-(-7)}{0-1} = -9$$

Since

$$\text{Gradient of } AB \times \text{Gradient of } AC = \frac{1}{9} \times -9 = -1$$

This implies that AB is perpendicular to AC. By the property, angles in a semicircle, BC is the hypotenuse and is the diameter of the circle

□

(ii)

$$\text{Midpoint of } BC = \left(\frac{9+1}{2}, \frac{3-7}{2}\right) = (5, -2)$$

$$\text{Radius of circle} = \sqrt{(9-5)^2 + (3+2)^2} = \sqrt{41}$$

$$\therefore \boldsymbol{(x-5)^2 + (y+2)^2 = 41}$$

4. (a)

$$\text{Midpoint of } AB = \left(\frac{11-1}{2}, \frac{13+7}{2}\right) = (5, 10)$$

$$\text{Radius} = \sqrt{(11-5)^2 + (13-10)^2} = \sqrt{45}$$

$$\therefore \boldsymbol{(x-5)^2 + (y-10)^2 = 45}$$

(b)

$$\text{Gradient of } QF = \frac{10-4}{5-2} = 2$$

$$\therefore \text{Gradient of tangent} = -\frac{1}{2}$$

Equation of tangent:

$$y - 4 = -\frac{1}{2}(x-2)$$
$$\boldsymbol{y = -\frac{1}{2}x + 5}$$

(c)

$$\text{Radius of } C_2 = \sqrt{(1-2)^2 + (2-4)^2} = \sqrt{5}$$

$$\text{Distance from } P \text{ to the point} = 3 > \sqrt{5}$$

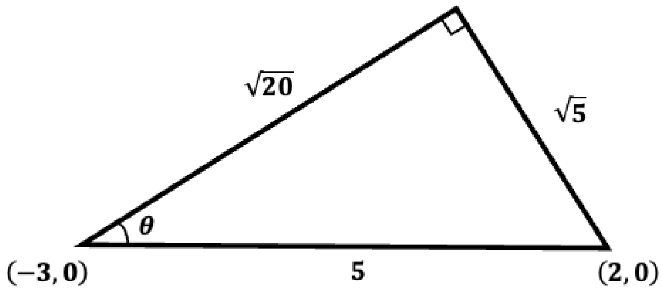
Since the distance from the centre P to the point is greater than the radius of C_2, the point lies **outside** of C_2

(d) **Yes**, origin O lies on C_3. Note that the x and y-axes are perpendicular to each other, and since DE is the diameter of the circle, by property: angles in a semicircle, origin O lies on C_3

5. (a) From the equation of the circle,

$$\text{centre} = (2, 0) \qquad \text{radius} = \sqrt{5}$$

A right-triangle can be drawn with the above information of the question



$$\begin{aligned} \therefore \text{Gradient of tangent line} &= \pm \tan\theta \\ &= \pm\frac{\sqrt{5}}{\sqrt{20}} \\ &= \pm\frac{1}{2} \end{aligned}$$

Hence,

$$y = -\frac{1}{2}(x+3) \qquad \text{or} \qquad y = \frac{1}{2}(x+3)$$
$$\boldsymbol{y = -\frac{1}{2}x - \frac{3}{2}} \qquad \text{or} \qquad \boldsymbol{y = \frac{1}{2}x + \frac{3}{2}}$$

(b)

$$(x-2)^2 + y^2 = 5 \text{ ......(1)} \qquad y = \pm\frac{1}{2}x \pm \frac{3}{2} \text{ ......(2)}$$

Substitute Equation (2) into Equation (1),

$$\begin{aligned} (x-2)^2 + \left(\pm\frac{1}{2}x \pm \frac{3}{2}\right)^2 &= 5 \\ x^2 - 4x + 4 + \frac{1}{4}x^2 + \frac{3}{2}x + \frac{9}{4} - 5 &= 0 \\ 5x^2 - 10x - 5 &= 0 \\ x^2 - 2x - 1 &= 0 \\ (x-1)^2 &= 0 \\ x &= 1 \end{aligned}$$

Substituting $x = 1$ into Equation (2),

$$y = \pm 2$$

$$\therefore \mathbf{(1, 2)} \qquad \mathbf{(1, -2)}$$

(c) **2 points of intersection**. Since the 2 tangent lines have a gradient of $\pm\frac{1}{2}$ respectively, any line that has a gradient between that range will intersect the graph at 2 distinct points

$$-\frac{1}{2} < \frac{1}{4} < \frac{1}{2}$$

10 Linear Law

10.1 Full Solutions

1. (a) From the equation

$$y\sqrt{x} = 5x - k$$

Since the y-intercept is -8,

$$k = \mathbf{-8}$$

Since the gradient is 5,

$$\begin{aligned} \tan\theta &= 5 \\ \theta &= \tan^{-1} 5 \\ &= 78.690067... \\ &= \mathbf{78.7^\circ\ (1.d.p.)} \end{aligned}$$

(b) When $y\sqrt{x} = 0$,

$$\begin{aligned} 5x - 8 &= 0 \\ x &= 1\frac{3}{5} \end{aligned}$$

$$\therefore \boldsymbol{R}\left(\mathbf{1\frac{3}{5}, 0}\right)$$

2. (a)

$$\begin{aligned} ax^2 + ky^3 - 120 &= 0 \\ ky^3 &= 120 - ax^2 \\ y^3 &= -\frac{a}{k}x^2 + \frac{120}{k} \end{aligned}$$

Hence, we plot a graph of y^3 against x^2 where the gradient is $-\frac{a}{k}$ and the y^3-intercept is $\frac{120}{k}$

(b) Since the angle is 60°,

$$\begin{aligned} \text{Gradient} &= \tan 60^\circ \\ &= \sqrt{3} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{y} - 1 &= \sqrt{3}\left(x - \sqrt{3}\right) \\ \frac{1}{y} &= \sqrt{3}x - 2 \end{aligned}$$

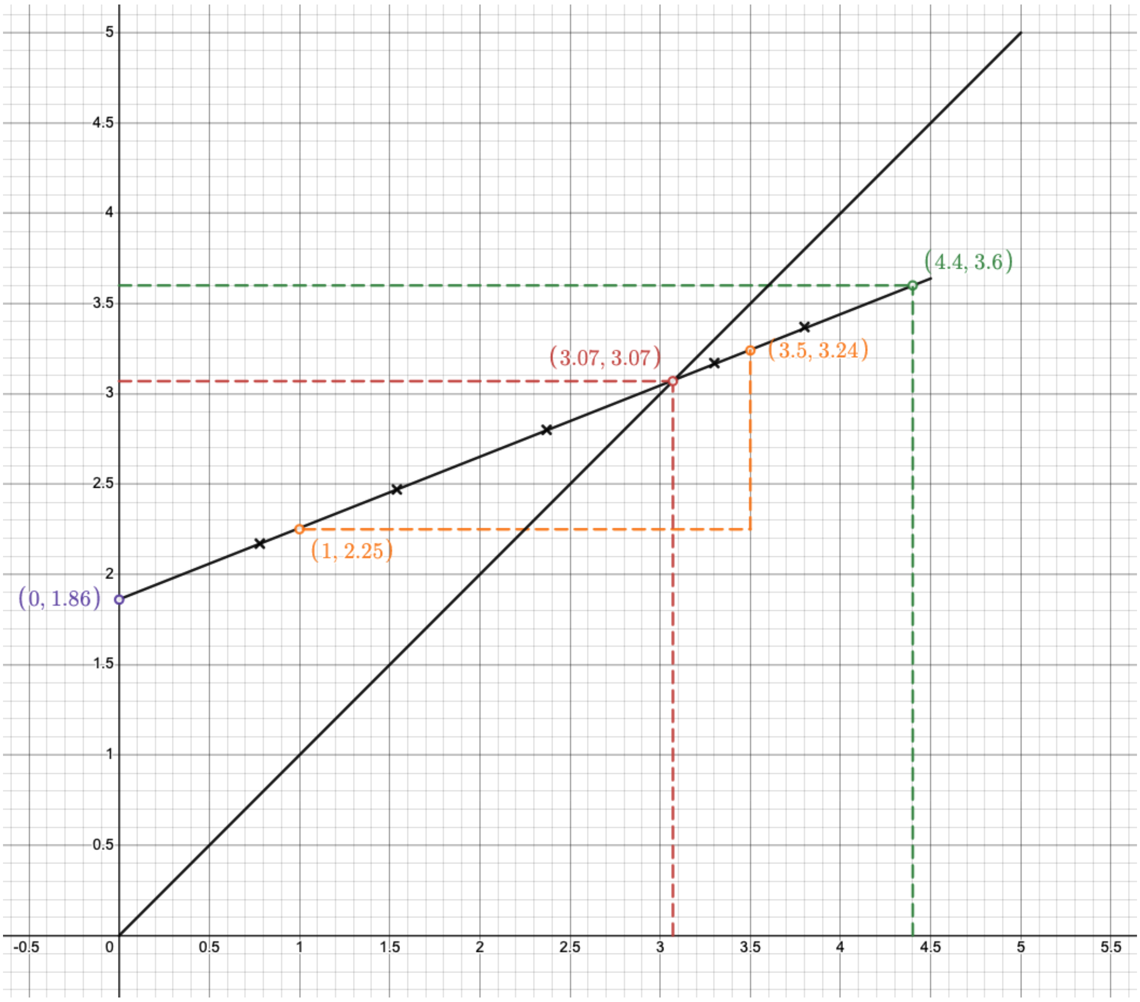
$$\therefore \boldsymbol{y} = \mathbf{\frac{1}{\sqrt{3} - 2}}$$

3. (a)

$$y = 10^k x^a$$
$$\lg y = a \lg x + k$$

Hence, we are plotting $\lg y$ against $\lg x$

$\lg x$	0.78	1.54	2.37	3.30	3.80
$\lg y$	2.17	2.47	2.80	3.17	3.37



(b) From the graph

$$a = \frac{3.24 - 2.25}{3.5 - 1}$$
$$= \mathbf{0.396\ (exact)}$$

$$k = \mathbf{1.86}$$

(c) When $y = 4000$, $\lg y = 3.60$ (3.s.f.)

$$\begin{aligned} \therefore \lg x &= 3.6 \\ x &= 10^{3.6} \\ &= \$3981.071706... \\ &= \mathbf{\$3981.07\ (2.d.p.)} \end{aligned}$$

(d) When the company breakeven, it is when the expenditure is equivalent to the sales generated

$$\text{Sketch the graph: } \lg y = \lg x$$

Hence, from the graph

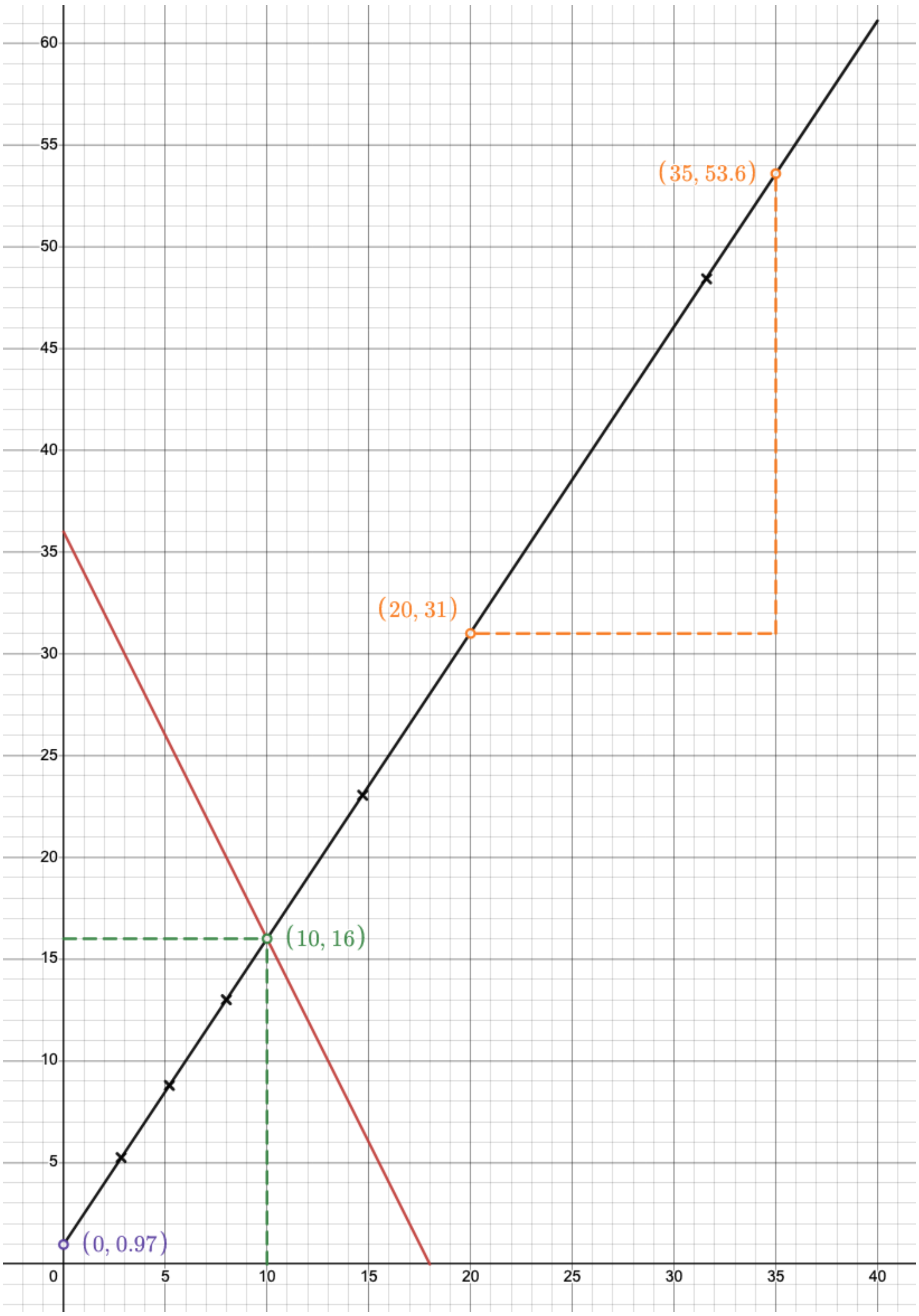
$$\begin{aligned} \lg x &= 3.07 \\ x &= 10^{3.07} \\ &= \$1174.897555.. \\ &= \mathbf{\$1174.90\ (2.d.p.)} \end{aligned}$$

4. (a)

$$\frac{y-b}{x} = a\sqrt{x} - 1$$
$$y - b = ax\sqrt{x} - x$$
$$(x+y) = a\left(x\sqrt{x}\right) + b$$

Hence, we are plotting $(x+y)$ against $x\sqrt{x}$

$x\sqrt{x}$	2.83	5.20	8	14.7	31.6
$x+y$	5.24	8.79	13	23.05	48.43



(b) From the graph

$$\begin{aligned} a &= \frac{53.6-31}{35-20} \\ &= \mathbf{1.51\ (2.d.p.)} \end{aligned}$$

$$b = \mathbf{0.97\ (2.d.p.)}$$

(c)

$$\begin{aligned} y + x + 2x\sqrt{x} &= 36 \\ (x+y) &= -2\left(x\sqrt{x}\right) + 36 \end{aligned}$$

Line (in red) is drawn on the previous page

(d)

$$\begin{aligned} (a+2)x\sqrt{x} &= 36 - b \\ ax\sqrt{x} + 2x\sqrt{x} &= 36 - b \\ ax\sqrt{x} + b &= -2x\sqrt{x} + 36 \end{aligned}$$

Hence, we are looking for the point of intersection between the 2 lines

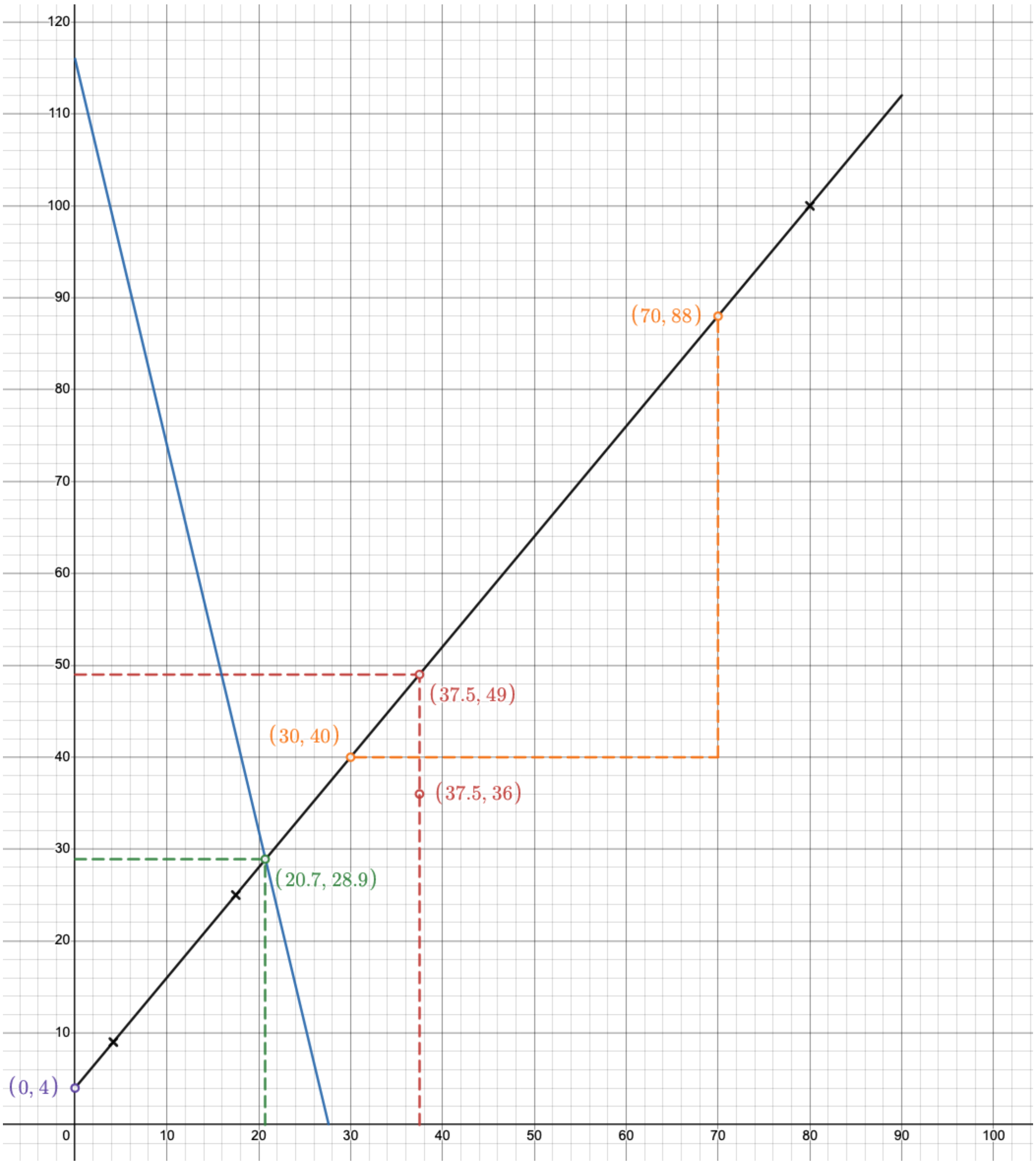
$$\begin{aligned} \therefore x\sqrt{x} &= 10 \\ x^{1.5} &= 10 \\ x &= \sqrt[1.5]{10} \\ &= 4.641588... \\ &= \mathbf{4.64\ (3.s.f.)} \end{aligned}$$

5. (a) (i)

$$v = \sqrt{e^p + 2as}$$
$$v^2 = (2a)s + e^p$$

Hence, we are plotting v^2 against s

s	4.17	17.5	37.5	80
v^2	9	25	36	100



(ii)

$$\text{Incorrect value of } v = \mathbf{6\ m/s}$$

(iii) From the graph

$$v^2 = 49$$
$$v = \mathbf{7\ m/s} \text{ (rej -ve)}$$

(b) From the graph

$$2a = \frac{88 - 40}{70 - 30}$$
$$a = \frac{1}{2}(1.2)$$
$$= \mathbf{0.6\ (exact)}$$

$$e^p = 4$$
$$p = \mathbf{\ln 4}$$

(c) It represents the **square of initial velocity of the electric toy car**

(d)

$$s = \frac{120 - 2e^p}{4a + 3}$$
$$4as + 3s = 120 - 2e^p$$
$$2as + e^p = (-2a - 3)s + 120 - e^p$$

Hence, we are sketching the line of:

$$v^2 = (-2a - 3)s + (120 - e^p)$$

To solve the equation, we are looking for the point of intersection

$$s = \mathbf{20.7\ m}$$

11 Proofs of Plane Geometry

11.1 Full Solutions

1. (a)

$$\begin{aligned}\angle BDC &= \angle ABD \text{ (alternate angles)}\\ &= \angle BCE \text{ (alternate segment theorem)}\end{aligned}$$

$$\therefore \angle BCE = \angle BDC \textbf{ (shown)}$$

□

(b)

$$\angle BCE = \angle BDC \text{ (from above part) (A)}$$

$$\angle CBE = \angle DBC \text{ (common angle) (A)}$$

By the **AA** similarity test, $\triangle BCE$ is similar to $\triangle BDC$

□

(c) From part (b)

$$\frac{AE}{CE} = \frac{AB}{CD}$$

$$\frac{AE}{\left(\frac{1}{3}AC\right)} = \frac{AB}{CD}$$

$$\therefore 3AE \times CD = AB \times AC \textbf{ (shown)}$$

□

2. (a)

$$\angle PBM + \angle PXM = 180^\circ \textbf{ (angles in opposite segment)}$$

(b)

$$\angle BRM = \angle PRX \text{ (common angle) (A)}$$

$$\angle XPR = \angle BMR \text{ (exterior angle of cyclic quadrilateral) (A)}$$

By the **AA** similarity test, $\triangle RBM$ is similar to $\triangle RXP$

□

(c) From part (b),

$$\frac{PR}{RX} = \frac{RM}{RB} \text{ ......(1)}$$

Since $\triangle QXA$ and $\triangle QPM$ are similar,

$$\frac{PQ}{XQ} = \frac{OM}{QA} \text{ ......(2)}$$

From the given statement,

$$\frac{PR}{RX} = \frac{PQ}{QX} \text{ ......(3)}$$

Substitute Equation (1) and (2) into Equation (3),

$$\frac{RM}{RB} = \frac{QM}{QA}$$

Since $QM = MR$ as M is the midpoint of QR (given)

$$RB = QA \textbf{ (shown)}$$

□

3. (a)

$$\angle ADC = \angle BDA \text{ (common angle) (A)}$$
$$\angle CAD = \angle ABD \text{ (alternate segment theorem) (A)}$$
$$\text{By the } \mathbf{AA} \text{ similarity test, } \triangle ADC \text{ is similar to } \triangle BDA$$

□

(b) From part (a),

$$\frac{BD}{AD} = \frac{AD}{CD}$$
$$BD \times CD = AD^2 \text{ ......(1)}$$

Since AD is tangential to the circle

$$\angle DAE = 90^\circ \text{ (tangent perpendicular to radius)}$$
$$\therefore AD^2 = DE^2 - AE^2 \text{ (pythagoras' theorem) ....(2)}$$

Let Equation (1) = Equation (2),

$$BD \times CD = DE^2 - AE^2 \textbf{ (shown)}$$

□

4. (a)

$$\angle CAF = \angle BAF \text{ (common angle) (A)}$$
$$\angle AFB = \angle ACF \text{ (alternate segment theorem) (A)}$$
$$\text{By the } \mathbf{AA} \text{ similarity test, } \triangle ABF \text{ is similar to } \triangle AFC$$

□

(b) From part (a),

$$\frac{CF}{FB} = \frac{AC}{AF}$$
$$CF \times AF = AC \times FB \textbf{ (shown)}$$

□

(c) Since $\triangle DEC$ and $\triangle BEF$ are similar

$$\frac{BE}{DE} = \frac{EF}{EC}$$
$$BE \times EC = ED \times EF$$
$$\frac{2}{5}BC \times \frac{3}{5}BC = ED \times EF$$
$$\therefore EF \times ED = \frac{6}{25}BC^2 \textbf{ (shown)}$$

□

12 Differentiation

12.1 Full Solutions

1. (a)

$$a = \mathbf{\frac{1}{4}}$$

(b)

$$\begin{aligned}\frac{dy}{dx} &= \frac{(4x-1)(9)-(9x-3b)(4)}{(4x-1)^2}\\ &= \frac{36x-9-36x+12b}{(4x-1)^2}\\ &= \frac{12b-9}{(4x-1)^2}\end{aligned}$$

Since y is an increasing function,

$$\frac{dy}{dx} > 0$$

Since $(4x-1)^2 > 0$

$$12b - 9 > 0$$

$$\therefore \boldsymbol{b > \frac{3}{4}}$$

(c)

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx}\\ &= \frac{12}{\left(\frac{dt}{dx}\right)} \times \frac{dt}{dx}\\ &= 12\end{aligned}$$

Hence,

$$\begin{aligned}\frac{12(3)-9}{(4x-1)^2} &= 12\\ (4x-1)^2 &= \frac{9}{4}\\ 4x-1 &= \pm\frac{3}{2} \quad \text{(rej -ve)}\end{aligned}$$

$$\therefore x = \mathbf{\frac{5}{8}}$$

2. (a)

$$V = 4\sin \pi t$$
$$\frac{dV}{dt} = 4\pi \cos \pi t$$

$$\therefore \left.\frac{dV}{dt}\right|_{x=\frac{1}{4}} = 4\pi \cos \pi \left(\frac{1}{4}\right)$$
$$= \boldsymbol{\frac{4\pi}{\sqrt{2}}} \textbf{ v/s}$$

(b)

$$\frac{dV}{dt} = 2\pi\sqrt{3}$$
$$\cos \pi t = \frac{\sqrt{3}}{2}$$

$$\text{Basic angle } \alpha = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi}{6} \quad \text{(Quadrant 1 and 4)}$$

$$\pi t = \frac{\pi}{6} \qquad \pi t = 2\pi - \frac{\pi}{6} \qquad \pi t = 2\pi + \frac{\pi}{6} \qquad \pi t = 4\pi - \frac{\pi}{6}$$
$$t = \boldsymbol{\frac{1}{6}} \textbf{ sec} \qquad t = \boldsymbol{1\frac{5}{6}} \textbf{ sec} \qquad t = \boldsymbol{2\frac{1}{6}} \textbf{ sec} \qquad t = \boldsymbol{3\frac{5}{6}} \textbf{ sec}$$

(c)

$$\frac{dI}{dV} = \frac{1}{5}$$

$$\frac{dI}{dt} = \frac{dI}{dV} \times \frac{dV}{dt}$$
$$= \left(\frac{1}{5}\right) \times 2$$
$$= \boldsymbol{\frac{2}{5}} \textbf{ amperes/sec}$$

3. (a) We are **unable to take the natural logarithm of any negative number**, hence $x < 4$ condition is necessary

(b)

$$f(x) = \frac{\ln(4-x)}{x-4}$$

$$f'(x) = \frac{(x-4)\left(-\frac{1}{4-x}\right) - (1)\ln(4-x)}{(x-4)^2}$$

$$= \mathbf{\frac{1-\ln(4-x)}{(x-4)^2}}$$

(c) For $f(x)$ to be decreasing,

$$f'(x) < 0$$

Since $(x-4)^2 > 0$

$$1 - \ln(4-x) < 0$$
$$\ln(4-x) > 1$$
$$4 - x > e$$

$$x < 4 - e \ \textbf{(shown)}$$

□

4. (a)

$$\begin{aligned} f(x) &= x^3 \ln x \\ f'(x) &= x^3 \left(\frac{1}{x}\right) + 3x^2 \ln x \\ &= x^2 (1 + 3\ln x) \end{aligned}$$

At stationary point, $f'(x) = 0$

$$x^2 = 0 \text{ (no solution)} \qquad \text{or} \qquad 1 + 3\ln x = 0$$

$$\therefore \textbf{1 solution only}$$

(b)

$$\begin{aligned} 1 + 3\ln x &= 0 \\ \ln x &= -\frac{1}{3} \\ x &= e^{-\frac{1}{3}} \\ &= \boldsymbol{\frac{1}{\sqrt[3]{e}}} \end{aligned}$$

(c)

$$\begin{aligned} f'(x) &= x^2 (1 + 3\ln x) \\ f''(x) &= 2x (1 + 3\ln x) + x^2 \left(\frac{3}{x}\right) \\ &= x [2 + 6\ln x + 3] \\ &= x (5 + 6\ln x) \end{aligned}$$

Hence,

$$\begin{aligned} f''\left(\frac{1}{\sqrt[3]{e}}\right) &= \left(\frac{1}{\sqrt[3]{e}}\right)\left[5 + 6\ln\left(\frac{1}{\sqrt[3]{e}}\right)\right] \\ &= \left(\frac{1}{\sqrt[3]{e}}\right)\left[5 + \frac{6}{(-3)}\right] \\ &= \frac{3}{\sqrt[3]{e}} \ \textbf{(shown)} \end{aligned}$$

□

(d) Since $f''(x) > 0$, the stationary point is a **minimum point**

5. (a) By Pythagoras' Theorem,

$$\begin{aligned}(h+x)^2+(h+x)^2 &= \left(4\sqrt{2}\right)^2\\ (h+x)^2 &= 16\\ h+x &= \pm 4 \text{ (rej -ve)}\\ h &= 4-x \textbf{ (shown)}\end{aligned}$$

□

(b) Let the height of the pyramid be y

$$\begin{aligned}y^2+x^2 &= (4-x)^2\\ y^2 &= 16-8x\\ y &= \sqrt{16-8x} \text{ (rej -ve)}\end{aligned}$$

$$\begin{aligned}V &= \frac{1}{3}(2x)^2y\\ &= \frac{1}{3}\left(4x^2\right)\sqrt{16-8x}\\ &= \frac{8}{3}x^2\sqrt{4-2x} \textbf{ (shown)}\end{aligned}$$

□

(c)

$$\begin{aligned}\frac{dV}{dx} &= \frac{8}{3}\left\{x^2\left[\frac{1}{2}(4-2x)^{-\frac{1}{2}}(-2)\right]+2x\sqrt{4-2x}\right\}\\ &= \frac{8}{3}\left[2x\sqrt{4-2x}-\frac{x^2}{\sqrt{4-2x}}\right]\\ &= \frac{8}{3}x\left[\frac{2(4-2x)-x}{\sqrt{4-2x}}\right]\end{aligned}$$

At the maximum,

$$\begin{aligned}\frac{dV}{dx} &= 0\\ 2(4-2x)-x &= 0\\ x &= 1\frac{3}{5} \text{ m}\end{aligned}$$

Hence,

$$\begin{aligned}V|_{x=1\frac{3}{5}} &= \frac{8}{3}\left(1\frac{3}{5}\right)^2\sqrt{4-2\left(1\frac{3}{5}\right)}\\ &= 6.105956...\\ &= \mathbf{6.11\ m^3\ (3.s.f.)}\end{aligned}$$

6. (a)

$$y = (x+k)^2$$
$$\frac{dy}{dx} = 2(x+k)$$

Gradient of tangent:

$$\left.\frac{dy}{dx}\right|_{x=2k} = 2(2k+k)$$
$$= 6k$$

When $x = 2k$,

$$y = (2k+k)^2$$
$$= 9k^2$$

Equation of tangent:

$$y - 9k^2 = 6k\,(x - 2k)$$

$$\therefore \boldsymbol{y = 6kx - 3k^2}$$

(b) At P, $y = 0$

$$6kx = 3k^2$$
$$x = \frac{k}{2}$$

$$\therefore P\left(\frac{k}{2}, 0\right)$$

At Q, $x = 0$,

$$y = 6k(0) - 3k^2$$
$$= -3k^2$$

$$\therefore Q\left(0, -3k^2\right)$$

$$\text{Midpoint } M = \left(\frac{\left(\frac{k}{2}\right)+0}{2}, \frac{0-3k^2}{2}\right)$$
$$= \left(\frac{k}{4}, -\frac{3k^2}{2}\right)$$

Substitute M into the curve,

$$\left(-\frac{3k^2}{2}\right) + 24\left(\frac{k}{4}\right)^2 = -\frac{3k^2}{2} + \frac{3k^2}{2}$$
$$= 0$$

$\therefore$ Hence, M lies on the curve

□

13 Integration

13.1 Full Solutions

1.

$$\frac{d^2y}{dx^2} = 16e^{-4x}$$

$$\begin{aligned}\frac{dy}{dx} &= \int 16e^{-4x}\ dx \\ &= -4e^{-4x} + c\end{aligned}$$

To find the arbitrary constant c,

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=0} &= 3 \\ -4e^{-4(0)} + c &= 3 \\ c &= 7\end{aligned}$$

$$\therefore \frac{dy}{dx} = -4e^{-4x} + 7$$

$$\begin{aligned}y &= \int \left(-4e^{-4x} + 7\right)\ dx \\ &= e^{-4x} + 7x + d\end{aligned}$$

To find the arbitrary constant d, substitute $\left(2, e^{-8}\right)$ into the curve y

$$\begin{aligned}e^{-8} &= e^{-4(2)} + 7(2) + d \\ d &= -14\end{aligned}$$

$$\therefore \boldsymbol{y = e^{-4x} + 7x - 14}$$

2. (a)

$$\begin{aligned}\int_{-1}^{3} f(x)\ dx &= \int_{-1}^{3} [f(x) + 1]\ dx - \int_{-1}^{3} 1\ dx \\ &= 8 - [x]_{-1}^{3} \\ &= \mathbf{4}\end{aligned}$$

(b)

$$\begin{aligned}\int_{2}^{3} [f(x) + 1]\ dx - \int_{2}^{-1} [f(x) + 1]\ dx &= \int_{2}^{3} [f(x) + 1]\ dx + \int_{-1}^{2} [f(x) + 1]\ dx \\ &= \int_{-1}^{3} [f(x) + 1]\ dx \\ &= \mathbf{8}\end{aligned}$$

3. (a)

$$\frac{dy}{dx} = \frac{2}{x+5}$$

When $x = -1$,

$$\left.\frac{dy}{dx}\right|_{x=-1} = \frac{2}{-1+5} = \frac{1}{2}$$

$$\text{Gradient of Normal} = -2$$

When $x = -1$,

$$\begin{aligned} y &= 2\ln(-1+5) \\ &= 2\ln 4 \\ &= 4\ln 2 \end{aligned}$$

Hence,

$$\begin{aligned} y - 4\ln 2 &= -2\,(x+1) \\ y &= -2x + 4\ln 2 - 2 \ \textbf{(shown)} \end{aligned}$$

□

(b) When the normal touches the x-axis, $y = 0$

$$\begin{aligned} 0 &= -2x + 4\ln 2 - 2 \\ x &= 2\ln 2 - 1 \end{aligned}$$

$$\begin{aligned} 2\ln(x+5) &= y \\ \ln(x+5) &= \frac{y}{2} \\ x &= e^{\frac{y}{2}} - 5 \end{aligned}$$

Hence,

$$\begin{aligned} \text{Shaded area} &= \left|\int_0^{4\ln 2} e^{\frac{y}{2}} - 5 \ dy\right| - \Big\{|-1| \times (4\ln 2)\Big\} + \frac{1}{2}(4\ln 2)\,[2\ln 2 - 1 + 1] \\ &= \left|\Big[2e^{\frac{y}{2}} - 5y\Big]_0^{4\ln 2}\right| - 4\ln 2 + (2\ln 2)^2 \\ &= 7.01216... \\ &= \textbf{7.01 units}^{\mathbf{2}} \end{aligned}$$

4. (a)

$$\text{Radius of } C_1 = \textbf{4 units}$$

(b)

$$\text{Equation of circle } C_1 : \boldsymbol{(x-4)^2 + (y+6)^2 = 16}$$

(c)

$$\begin{aligned}\text{Area} &= \int_{-2}^{0} y^2 \ dy + \left[4^2 - \frac{1}{4}\pi(4)^2\right] \\ &= \left[\frac{y^3}{3}\right]_{-2}^{0} + 16 - 4\pi \\ &= \left[0 - \left(-2\frac{2}{3}\right)\right] + 16 - 4\pi \\ &= 6.100296... \\ &= \textbf{6.10 units}^{\mathbf{2}} \ \textbf{(3.s.f.)}\end{aligned}$$

(d)

$$\text{Centre of } C_2 = (4, 10)$$

$$(x-4)^2 + (y-10)^2 = 16$$

$$\therefore \boldsymbol{x^2 + y^2 - 8x - 20y + 100 = 0}$$

5. (a) For the integral to be undefined,

$$\begin{aligned}2x^2 - x - 6 &= 0 \\ (x-2)(2x+3) &= 0\end{aligned}$$

$$\therefore x = \mathbf{2} \qquad \text{or} \qquad x = -\frac{\mathbf{3}}{\mathbf{2}}$$

(b)

$$\begin{aligned}\int_{m}^{6} \frac{x-2}{(x-2)(2x+3)} \ dx &= \frac{1}{2}\ln\frac{5}{3} \\ \int_{m}^{6} \frac{1}{2x+3} \ dx &= \frac{1}{2}\ln\frac{5}{3} \\ \frac{1}{2}\int_{m}^{6} \frac{2}{2x+3} &= \frac{1}{2}\ln\frac{5}{3}\end{aligned}$$

$$\begin{aligned}\therefore [\ln(2x+3)]_m^6 &= \ln\frac{5}{3} \\ \ln\left(\frac{15}{2m+3}\right) &= \ln\frac{5}{3}\end{aligned}$$

By symmetry,

$$\begin{aligned}\frac{2m+3}{3} &= 3 \\ m &= \mathbf{3}\end{aligned}$$

14 Differentiation & Integration

14.1 Full Solutions

1. (a)

$$\begin{aligned} y &= x^2 \ln x^3 \\ &= 3x^2 \ln x \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 6x \ln x + 3x^2 \left(\frac{1}{x}\right) \\ &= 3x\,(1 + 2\ln x) \quad \textbf{(shown)} \end{aligned}$$

□

(b) (i) At P, $y = 0$

$$x \ln x = 0$$

$$x = 0 \text{ (N.A.)} \qquad \text{or} \qquad \ln x = 0$$

Hence,

$$\begin{aligned} x &= e^0 \\ &= \mathbf{1} \end{aligned}$$

(ii)

$$\begin{aligned} \int_1^e 3x(1 + 2\ln x)\ dx &= \left[x^2 \ln x^3\right]_1^e \\ \int_1^e 3x + 6x \ln x\ dx &= e^2 \ln e^3 \\ \frac{3}{2}\left[x^2\right]_1^e + \int_1^e 6x \ln x\ dx &= 3e^2 \end{aligned}$$

$$\begin{aligned} \therefore \int_1^e 6x \ln x\ dx &= 3e^2 - \frac{3}{2}\left(e^2 - 1\right) \\ &= \frac{3}{2}\left(e^2 + 1\right) \end{aligned}$$

$$\begin{aligned} \therefore \text{Area} &= \int_1^e x \ln x\ dx \\ &= \frac{1}{4}\left(e^2 + 1\right) \quad \textbf{(shown)} \end{aligned}$$

□

2.

$$f'(x) = \sin 2x + \cos 3x$$

$$\begin{aligned} f(x) &= \int (\sin 2x + \cos 3x) \ dx \\ &= -\frac{1}{2}\cos 2x + \frac{1}{3}\sin 3x + c \end{aligned}$$

To find the arbitrary constant c,

$$\begin{aligned} f\left(\frac{\pi}{6}\right) &= -\frac{1}{2}\cos 2\left(\frac{\pi}{6}\right) + \frac{1}{3}\sin 3\left(\frac{\pi}{6}\right) + c \\ c &= -\frac{1}{12} \\ \therefore f(x) &= -\frac{1}{2}\cos 2x + \frac{1}{3}\sin 3x - \frac{1}{12} \\ f''(x) &= 2\cos 2x - 3\sin 3x \end{aligned}$$

Hence,

$$\begin{aligned} f''(x) + 9f(x) &= 2\cos 2x - 3\sin 3x + 9\left(-\frac{1}{2}\cos 2x + \frac{1}{3}\sin 3x - \frac{1}{12}\right) \\ &= 2\cos 2x - 3\sin 3x - \frac{9}{2}\cos 2x + 3\sin 3x - \frac{3}{4} \\ &= -\frac{3}{4} - \frac{5}{2}\cos 2x \ \textbf{(shown)} \end{aligned}$$

□

3.

$$f'(x) = \frac{1}{2x-5} - \frac{4}{(2x-5)^2}$$

$$\begin{aligned} f(x) &= \int \frac{1}{2x-5} - \frac{4}{(2x-5)^2} \ dx \\ &= \frac{1}{2}\ln(2x-5) + \frac{2}{2x-5} + c \end{aligned}$$

To find the arbitrary constant c,

$$\begin{aligned} f(3) &= \frac{1}{2}\ln\left[2(3)-5\right] + \frac{2}{2(3)-5} + c \\ c &= -\frac{1}{4} \\ \therefore f(x) &= \frac{1}{2}\ln(2x-5) + \frac{2}{2x-5} - \frac{1}{4} \\ f''(x) &= -\frac{2}{(2x-5)^2} + \frac{16}{(2x-5)^3} \end{aligned}$$

Hence,

$$\begin{aligned} 8f(x) - (2x-5)^2 f''(x) &= 8\left[\frac{1}{2}\ln(2x-5) + \frac{2}{2x-5} - \frac{1}{4}\right] - (2x-5)^2\left[-\frac{2}{(2x-5)^2} + \frac{16}{(2x-5)^3}\right] \\ &= 4\ln(2x-5) + \frac{16}{2x-5} - 2 + 2 - \frac{16}{2x-5} \\ &= \ln(2x-5)^4 \ \textbf{(shown)} \end{aligned}$$

□

4. (a)

$$y = x^3 e^{1-2x}$$

$$\begin{aligned}\frac{dy}{dx} &= x^3 e^{1-2x}(-2) + 3x^2 e^{1-2x} \\ &= x^2 e^{1-2x}(3-2x)\end{aligned}$$

For the function to be decreasing,

$$\begin{aligned}\frac{dy}{dx} &< 0 \\ x^2 e^{1-2x}(3-2x) &< 0\end{aligned}$$

Note that

$$x^2 e^{1-2x} > 0$$

$$\therefore 3 - 2x < 0$$

$$\boldsymbol{x > 1\frac{1}{2}}$$

(b) (i)

$$y = [\ln(3-4x)]^2$$

$$\begin{aligned}\frac{dy}{dx} &= 2\ln(3-4x)\left(-\frac{4}{3-4x}\right) \\ &= \frac{-8\ln(3-4x)}{3-4x}\end{aligned}$$

$$\therefore k = \mathbf{8}$$

(ii)

$$\begin{aligned}\int_{-2}^{-1} \frac{2+3\ln(3-4x)}{3-4x}\,dx &= \int_{-2}^{-1} \frac{2}{3-4x}\,dx - \frac{3}{8}\int_{-2}^{-1} \frac{-8\ln(3-4x)}{3-4x}\,dx \\ &= -\frac{1}{2}\left[\ln(3-4x)\right]_{-2}^{-1} - \frac{3}{8}\left[\ln(3-4x)^2\right]_{-2}^{-1} \\ &= -\frac{1}{2}\left(\ln 7 - \ln 11\right) - \frac{3}{8}\left[(\ln 7)^2 - (\ln 11)^2\right] \\ &= 0.96224... \\ &= \mathbf{0.962\ (3.s.f.)}\end{aligned}$$

5. (a)

$$\begin{aligned}\frac{d}{dx}\left[(x-1)\sqrt{5+4x}\right] &= (1)\sqrt{5+4x}+(x-1)\left[\frac{1}{2}(5+4x)^{-\frac{1}{2}}\cdot(4)\right]\\ &= \sqrt{5+4x}+\frac{2(x-1)}{\sqrt{5+4x}}\\ &= \frac{5+4x+2(x-1)}{\sqrt{5+4x}}\\ &= \frac{6x+3}{\sqrt{5+4x}}\ \textbf{(shown)}\end{aligned}$$

□

(b) (i) At P, $x=1$

$$\begin{aligned}y|_{x=1} &= \frac{2(1)+1}{\sqrt{5+4(1)}}\\ &= 1\end{aligned}$$

$$\therefore \text{Gradient of normal} = -1$$

Equation of normal:

$$y-1=-(x-1)$$

$$\therefore \boldsymbol{y=-x+2}$$

(ii)

$$\begin{aligned}\text{Area of trapezium}PQOS &= \frac{1}{2}\left[OQ+PS\right](OS)\\ &= \frac{1}{2}\left[\left(\frac{2(0)+1}{\sqrt{5+4(0)}}\right)+\left(\frac{2(1)+1}{\sqrt{5+4(1)}}\right)\right](1)\\ &= \frac{1}{2}\left(1+\frac{1}{\sqrt{5}}\right)\end{aligned}$$

$$\text{Area under the curve from 0 to 1} = \int_0^1 \frac{2x+1}{\sqrt{5+4x}}\,dx$$

From the diagram, it is clear that the area of the trapezium is smaller than the area under the curve. Hence, it is safe to conclude that

$$\therefore \int_0^1 \frac{2x+1}{\sqrt{5+4x}}\,dx > \frac{1}{2}\left(1+\frac{1}{\sqrt{5}}\right)\ \textbf{(shown)}$$

□

(iii) At R, $y=0$

$$\begin{aligned}0 &= -x+2\\ x &= 2\end{aligned}$$

$$\begin{aligned}\text{Area of shaded region} &= \int_0^1 \frac{2x+1}{\sqrt{5+4x}}\,dx+\frac{1}{2}(1)(2)\\ &= \frac{1}{3}\int_0^1 \frac{6x+3}{\sqrt{5x+4}}\,dx+1\\ &= \frac{1}{3}\left[(x-1)\sqrt{5-4x}\right]_0^1+1\\ &= \left(\boldsymbol{\frac{1}{3}\sqrt{5}+1}\right)\ \textbf{units}^{\mathbf{2}}\end{aligned}$$

15 Kinematics

15.1 Full Solutions

1. (a)

$$a = -4e^{-t}$$

$$\begin{aligned} v &= \int -4e^{-t}\ dt \\ &= 4e^{-t} + c \end{aligned}$$

To find the arbitrary constant c, when $t = 0$, $v = 2$

$$\begin{aligned} 3 &= 4e^{-(0)} + c \\ c &= -2 \end{aligned}$$

$$\therefore v = 4e^{-t} - 2$$

At instantaneous rest, $v = 0$,

$$\begin{aligned} 4e^{-t} &= 2 \\ e^{-t} &= \frac{1}{2} \\ \therefore t &= -\ln\frac{1}{2}\ \textbf{(shown)} \end{aligned}$$

□

(b)

$$\begin{aligned} s &= \int 4e^{-t} - 2\ dt \\ &= -4e^{-t} - 2t + d \end{aligned}$$

To find the arbitrary constant d, when $t = 0$, $s = 0$

$$\begin{aligned} 0 &= -4e^{0} - 2(0) + d \\ d &= 4 \end{aligned}$$

$$\therefore s = -4e^{-t} - 2t + 4$$

Distance from instantaneous rest:

$$\begin{aligned} \text{Distance} &= -4e^{-\left(-\ln\frac{1}{2}\right)} - 2\left(-\ln\frac{1}{2}\right) + 4 \\ &= 2 + 2\ln\frac{1}{2} \end{aligned}$$

Distance from instantaneous rest to 2 sec:

$$\begin{aligned} \text{Distance} &= \left(2 + 2\ln\frac{1}{2}\right) - \left[-4e^{-(2)} - 2(2) + 4\right] \\ &= 2 + 2\ln\frac{1}{2} + \frac{4}{e^2} \end{aligned}$$

$$\begin{aligned} \therefore \text{Total distance travelled} &= \left(2 + 2\ln\frac{1}{2}\right) + \left(2 + 2\ln\frac{1}{2} + \frac{4}{e^2}\right) \\ &= 1.76876... \\ &= \mathbf{1.77\ m\ (3.s.f.)} \end{aligned}$$

(c)

$$\begin{aligned}\text{Average velocity} &= \frac{\text{Total displacement}}{\text{Total time}} \\ &= \frac{\left(-\frac{4}{e^2}\right)}{2} \\ &= -0.270670... \\ &= \mathbf{-0.271\ m/s}\end{aligned}$$

2. (a)

$$\begin{aligned}v &= 2\cos^2 t - 1 \\ &= \cos 2t\end{aligned}$$

$$\begin{aligned}a &= \frac{dv}{dt} \\ &= -2\sin 2t\end{aligned}$$

Hence, when $t = 2$,

$$\begin{aligned}a\big|_{t=2} &= -2\sin 2(2) \\ &= 1.513604... \\ &= \mathbf{1.51\ m/s^2\ (3.s.f.)}\end{aligned}$$

(b) When the particle is at instantaneous rest, $v = 0$

$$\begin{aligned}\cos 2t &= 0 \\ \alpha &= \frac{\pi}{4} \text{ (Quadrant 1 and 4)}\end{aligned}$$

Hence,

$$\begin{aligned}2t &= \frac{\pi}{4} \\ t &= \boldsymbol{\frac{\pi}{8}} \textbf{ sec}\end{aligned}$$

(c)

$$v = \cos 2t$$

$$\begin{aligned}s &= \int v\ dt \\ &= \frac{1}{2}\sin 2t + c\end{aligned}$$

To find the arbitrary constant c, when $t = 0$, $s = 0$

$$\begin{aligned}0 &= \frac{1}{2}\sin 2(0) + c \\ c &= 0\end{aligned}$$

$$s = \frac{1}{2}\sin 2t$$

$$\begin{aligned}\text{Total distance travelled} &= 2\left[\frac{1}{2}\sin 2\left(\frac{\pi}{8}\right)\right] + \left|\frac{1}{2}\sin 2(2)\right| \\ &= 1.085508... \\ &= \mathbf{1.08\ m\ (3.s.f.)}\end{aligned}$$

3. (a)

$$v = 3t^2 + pt + q$$

$$\begin{aligned} s &= \int v \ dt \\ &= t^3 + \frac{1}{2}pt^2 + qt + c \end{aligned}$$

To find the arbitrary constant c, when $t = 0$, $s = 3$

$$\begin{aligned} 3 &= (0)^3 + \frac{1}{2}p(0)^2 + q(0) + c \\ c &= 3 \end{aligned}$$

$$s = t^3 + \frac{1}{2}pt^2 + qt + 3$$

When $t = 2$, $s = 23$

$$\begin{aligned} 23 &= (2)^3 + \frac{1}{2}p(2)^2 + q(2) + 3 \\ 2p + 2q &= 12 \\ p + q &= 6 \text{ ......(1)} \end{aligned}$$

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= 6t + p \end{aligned}$$

When $t = 2$,

$$\begin{aligned} -6 &= 6(2) + p \\ p &= \mathbf{-18} \end{aligned}$$

Hence, substitute $p = -18$ into Equation (1),

$$\begin{aligned} -18 + q &= 6 \\ q &= \mathbf{24} \end{aligned}$$

(b) For the particle to return to the starting point, $s = 3$

$$\begin{aligned} t^3 - 9t^2 + 24t + 3 &= 3 \\ t\left(t^2 - 9t + 24\right) &= 0 \end{aligned}$$

For the quadratic factor,

$$\begin{aligned} b^2 - 4ac &= (-9)^2 - 4(1)(24) \\ &= -15 < 0 \end{aligned}$$

Hence, the quadratic factor has no solutions. The only time that the particle will be at the starting position is when $t = 0$, which in this case is at the start of the journey. The particle will **never return back to the starting point**

4. (a) At initial velocity, $t = 0$

$$\begin{aligned} v &= 15 - e^{-3(0)} \\ &= \mathbf{14\ m/s} \end{aligned}$$

(b) When $t \to \infty$, $e^{-3t} \to 0$. Velocity will approach a maximum speed of 15 m/s and hold it constant.

(c)

$$v = 15 - e^{-3t}$$

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= 3e^{-3t} \end{aligned}$$

When $t = 3$,

$$\begin{aligned} a &= \left[3e^{-3(3)}\right] \times 100 \\ &= 0.0370229... \\ &= \mathbf{0.037\ cm/s^2\ (3.d.p.)} \end{aligned}$$

(d) Lets check for any instantaneous rest, $v = 0$

$$\begin{aligned} e^{-3t} &= 15 \\ t &= -\frac{1}{3}\ln 15 \text{ (N.A.)} \end{aligned}$$

$$v = 15 - e^{-3t}$$

$$\begin{aligned} s &= \int v\ dt \\ &= 15t + \frac{1}{3}e^{-3t} + c \end{aligned}$$

When $t = 0$, $s = 0$

$$\begin{aligned} 0 &= 15(0) + \frac{1}{3}e^{-3(0)} + c \\ c &= -\frac{1}{3} \end{aligned}$$

$$\therefore s = 15t + \frac{1}{3}e^{-3t} - \frac{1}{3}$$

Hence, when $t = 4$,

$$\begin{aligned} s &= 15(4) + \frac{1}{3}e^{-3(4)} - \frac{1}{3} \\ &= 59.666668... \\ &= \mathbf{59.67\ m} \end{aligned}$$