Ong Kai Wen July 27, 2022

May Practice Questions 2022 Full Solutions (A-Math)

Copyright

Question Source

All questions are sourced and selected based on the known abilities of students sitting for the 'O' Level A-Math Examination. All questions compiled here are from **2018-2021 School Mid-Year / Prelim Papers**. Questions are categorised into the various topics and range in varying difficulties. If questions are sourced from respective sources, credit will be given when appropriate.

How to read:

[S4 ABCSS P1/2011 PRELIM Qn 1]

Secondary 4, ABC Secondary School, Paper 1, 2011, Prelim, Question 1

Syllabus (4049)

Algebra	**Geometry and Trigonometry**	**Calculus**
Quadratic Equations & Inequalities	Trigonometry	Differentiation
Surds	Coordinate Geometry	Integration
Polynomials	Further Coordinate Geometry	Kinematics
Simultaneous Equations	Linear Law	
Partial Fractions	Proofs of Plane Geometry	
Binomial Theorem		
Exponential & Logarithms		

Contents

1 Quadratic Equations & Inequalities

1.1 Full Solutions

1. (a)

$$\begin{aligned} 3^{2x+1} &= 6\left(3^{x-1}\right) - p \\ 3\left(3^{2x}\right) - 2\left(3^{x}\right) + p &= 0 \end{aligned}$$

Let $a = 3^x$,

$$3a^2 - 2a + p = 0$$

$$\begin{aligned} \text{Discriminant} &= (-2)^2 - 4(3)(p) \\ &= 4 - 12p \end{aligned}$$

Given that $p > \dfrac{1}{3}$,

$$\begin{aligned} -12p &< -4 \\ 4 - 12p &< 0 \end{aligned}$$

Since the discriminant is less than 0, the equation has no real solutions

□

(b)

$$y = 2x - \frac{a^2}{2} \text{ ......(1)}$$
$$y = x^2 - ax - 4 \text{ ......(2)}$$

Let Equation (1) = Equation (2),

$$\begin{aligned} x^2 - ax - 4 &= 2x - \frac{a^2}{2} \\ x^2 + (-a-2)x + \left(\frac{a^2}{2} - 4\right) &= 0 \end{aligned}$$

Since the line intersect the curve at 2 distinct points, $b^2 - 4ac > 0$

$$\begin{aligned} (-a-2)^2 - 4(1)\left(\frac{a^2}{2} - 4\right) &> 0 \\ -a^2 + 4a + 20 &> 0 \\ a^2 - 4a - 20 &< 0 \end{aligned}$$

Solving for a,

$$\begin{aligned} a &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-20)}}{2(1)} \\ &= \frac{4 \pm \sqrt{96}}{2} \\ &= 2 \pm 2\sqrt{6} \end{aligned}$$

$$\therefore \mathbf{2 - 2\sqrt{6} < a < 2 + 2\sqrt{6}}$$

2. (a)

$$y = px^2 - 4x + p$$

Since the curve lies entirely above the x-axis, $b^2 - 4ac < 0$

$$\begin{aligned}(-4)^2 - 4(p)(p) &< 0 \\ 4p^2 &> 16 \\ p^2 &> 4\end{aligned}$$

$$p < -2 \qquad \text{or} \qquad p > 2$$

Since the curve lies entirely above the x-axis, $p > 0$

$$\therefore \boldsymbol{p > 2}$$

(b)

$$y = x + 2k \text{ ......(1)}$$
$$2y^2 - x^2 = 8 \text{ ......(2)}$$

Substitute Equation (1) into Equation (2),

$$\begin{aligned}2(x + 2k)^2 - x^2 - 8 &= 0 \\ 2\left(x^2 + 4kx + 4k^2\right) - x^2 - 8 &= 0 \\ 2x^2 + 8kx + 8k^2 - x^2 - 8 &= 0 \\ x^2 + 8kx + \left(8k^2 - 8\right) &= 0\end{aligned}$$

To prove that the line will intersect the curve at 2 distinct points, WTS: $b^2 - 4ac > 0$

$$\begin{aligned}b^2 - 4ac &= (8k)^2 - 4(1)\left(8k^2 - 8\right) \\ &= 32k^2 + 32 \\ &= 32\left(k^2 + 1\right)\end{aligned}$$

Since for all real values of k,

$$\begin{aligned}k^2 &\geq 0 \\ k^2 + 1 &> 0 \\ 32\left(k^2 + 1\right) &> 0\end{aligned}$$

Since the discriminant is always positive for all real values of k, the line will intersect the curve at 2 distinct points

□

3. (a)

$$\begin{aligned} x^2 - x + 1 &= \left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 \\ &= \boldsymbol{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} \end{aligned}$$

(b) To show that the curve will cut the curve at 2 distinct points, WTS: $b^2 - 4ac > 0$

$$\begin{aligned} b^2 - 4ac &= (-2p)^2 - 4(1)(p-1) \\ &= 4p^2 - 4p + 4 \\ &= 4\left(p^2 - p + 1\right) \end{aligned}$$

From part (a),

$$\begin{aligned} b^2 - 4ac &= 4\left[\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}\right] \\ &= 4\left(x - \frac{1}{2}\right)^2 + 3 \end{aligned}$$

Since for all real values of p,

$$\begin{aligned} \left(x - \frac{1}{2}\right)^2 &\geq 0 \\ 4\left(x - \frac{1}{2}\right)^2 &\geq 0 \\ 4\left(x - \frac{1}{2}\right)^2 + 3 &> 0 \end{aligned}$$

Since the discriminant is always positive for all real values of p, the curve will cut the x-axis at 2 distinct points

$\square$

4. (a)

$$-\frac{4}{3x^2+14x-5} < 0$$

Since the fraction is always negative,

$$3x^2+14x-5 > 0$$
$$(3x-1)(x+5) > 0$$

$$\boldsymbol{x < -5 \qquad \text{and} \qquad x > \frac{1}{3}}$$

(b)

$$x+y = c \text{ ......(1)}$$
$$y^2 = 2x+3 \text{ ......(2)}$$

From part (1),

$$y = c-x \text{ ......(3)}$$

Substitute Equation (3) into Equation (2),

$$(c-x)^2 = 2x+3$$
$$x^2+(-2c-2)x+\left(c^2-3\right) = 0$$

Since the curve intersect the line at 2 distinct points,

$$(-2c-2)^2-4(1)\left(c^2-3\right) > 0$$
$$4c^2+8c+4-4c^2+12 > 0$$
$$8c+16 > 0$$

$$\boldsymbol{c > -2}$$

2 (Indices) and Surds

2.1 Full Solutions

1. (a)

$$\begin{aligned} 3^{n+2} - 3^n &= \frac{5^{n+1}}{25^n} \\ 9\left(3^n\right) - 3^n &= 5^{n+1-2n} \\ 8\left(3^n\right) &= \frac{5}{5^n} \\ \therefore 15^n &= \mathbf{\frac{5}{8}} \end{aligned}$$

(b)

$$\begin{aligned} x\sqrt{80} &= \sqrt{20} - x\sqrt{48} \\ x\left(\sqrt{80} + \sqrt{48}\right) &= \sqrt{20} \end{aligned}$$

$$\begin{aligned} \therefore x &= \frac{\sqrt{20}}{\sqrt{80} + \sqrt{48}} \\ &= \frac{2\sqrt{5}}{4\sqrt{5} + 4\sqrt{3}} \times \frac{4\sqrt{5} - 4\sqrt{3}}{4\sqrt{5} - 4\sqrt{3}} \\ &= \frac{40 - 8\sqrt{15}}{32} \\ &= \mathbf{\frac{5 - \sqrt{15}}{4}} \end{aligned}$$

2.

$$\begin{aligned} \text{Volume of prism} &= \frac{1}{2}\left(4 - \sqrt{5}\right)^2 (2)(h) \\ \left(50\sqrt{5} - 101\right) &= h\left(21 - 8\sqrt{5}\right) \end{aligned}$$

$$\begin{aligned} \therefore h &= \frac{50\sqrt{5} - 101}{21 - 8\sqrt{5}} \times \frac{21 + 8\sqrt{5}}{21 + 8\sqrt{5}} \\ &= \frac{1050\sqrt{5} + 2000 - 2121 - 808\sqrt{5}}{121} \\ &= \frac{242\sqrt{5} - 121}{121} \\ &= \mathbf{\left(2\sqrt{5} - 1\right)\ cm} \end{aligned}$$

3.

$$\text{Curved surface area of cone} = \pi r l$$

$$\begin{aligned}
\pi\left(5+2\sqrt{3}\right) l &= \left(51-3\sqrt{3}\right)\pi \\
l &= \frac{51-3\sqrt{3}}{5+2\sqrt{3}} \times \frac{5-2\sqrt{3}}{5-2\sqrt{3}} \\
&= \frac{255-102\sqrt{3}-15\sqrt{3}+18}{25-4(3)} \\
&= \frac{273-117\sqrt{3}}{13} \\
&= \left(\mathbf{21-9\sqrt{3}}\right)\ \textbf{cm}
\end{aligned}$$

4.

$$\begin{aligned}
\text{LHS} &= \frac{\sqrt{7}-\sqrt{6}}{\sqrt{21}+\sqrt{2}} \\
&= \frac{\sqrt{7}-\left(\sqrt{2}\right)\left(\sqrt{3}\right)}{\left(\sqrt{3}\right)\left(\sqrt{7}\right)+\sqrt{2}} \times \frac{\left(\sqrt{3}\right)\left(\sqrt{7}\right)-\sqrt{2}}{\left(\sqrt{3}\right)\left(\sqrt{7}\right)-\sqrt{2}} \\
&= \frac{7\sqrt{3}-\left(\sqrt{2}\right)\left(\sqrt{7}\right)-3\left(\sqrt{2}\right)\left(\sqrt{7}\right)+2\sqrt{3}}{19} \\
&= \frac{9}{19}\sqrt{3}-\frac{4}{19}\sqrt{14}
\end{aligned}$$

$$\therefore a = \mathbf{\frac{9}{19}} \qquad b = \mathbf{-\frac{4}{19}}$$

3 Polynomials

3.1 Full Solutions

1. (a) Let $x = -1$,

$$f(-1) = 9(-1)^3 - 6(-1)^2 - 11(-1) + 4$$
$$= 0$$

$$\therefore (x+1) \text{ is a factor of } f(x)$$

Let $x = \frac{4}{3}$,

$$f\left(\frac{4}{3}\right) = 9\left(\frac{4}{3}\right)^3 - 6\left(\frac{4}{3}\right)^2 - 11\left(\frac{4}{3}\right) + 4$$
$$= 0$$

$$\therefore (3x-4) \text{ is a factor of } f(x)$$

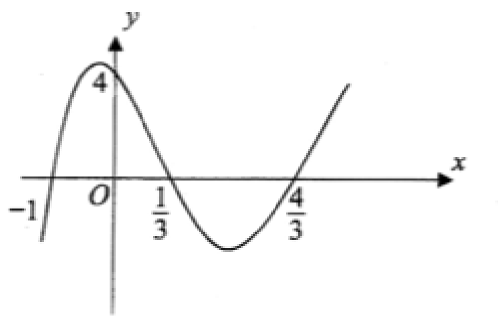
Let $x = \frac{1}{3}$,

$$f\left(\frac{1}{3}\right) = 9\left(\frac{1}{3}\right)^3 - 6\left(\frac{1}{3}\right)^2 - 11\left(\frac{1}{3}\right) + 4$$
$$= 0$$

$$\therefore (3x-1) \text{ is a factor of } f(x)$$

$$\therefore f(x) = \boldsymbol{(x+1)(3x-4)(3x-1)}$$

(b) Diagram



(c)

$$\boldsymbol{-1 \leq x \leq \frac{1}{3}, \quad x \geq \frac{4}{3}}$$

2. (a) Let a be an arbitrary constant

$$F(x) = a(x+1)(x-2)(x-5)$$

$$\therefore F(3) = 3$$

$$\therefore a(3+1)(3-2)(3-5) = 30$$
$$a = -\frac{15}{4}$$

$$\therefore F(x) = -\frac{15}{4}(x+1)(x-2)(x-5)$$

When divided by $(x+3)$,

$$F(-3) = -\frac{15}{4}(-3+1)(-3-2)(-3-5)$$
$$= \mathbf{300}$$

(b)

$$F\left(\sqrt{m}\right) = 0$$

$$-\frac{15}{4}\left(\sqrt{m}+1\right)\left(\sqrt{m}-2]\right)\left(\sqrt{m}-5\right) = 0$$

$$\sqrt{m} = -1 \text{ (N.A.)} \qquad \text{or} \qquad \sqrt{m} = 2 \qquad \text{or} \qquad \sqrt{m} = 5$$

$$\therefore m = \mathbf{4} \qquad \text{or} \qquad m = \mathbf{25}$$

3. (a) For $x^2 - 3x - 1 = 0$,

$$b^2 - 4ac = (-3)^2 - 4(1)(-1)$$
$$= 13 > 0$$

Since the discriminant of the factor is positive, there are 2 real roots

Hence, $f(x) = 0$ has 4 real solutions

□

(b)

$$f(x) = 3(x+2)(x-3)\left(x^2-3x-1\right)$$
$$= 3\left(x^2-x-6\right)\left(x^2-3x-1\right)$$
$$= \mathbf{3x^4 - 12x^3 - 12x^2 + 57x + 18}$$

(c) When divided by $(2x+1)$,

$$\text{Remainder} = 3\left(-\frac{1}{2}\right)^4 - 12\left(-\frac{1}{2}\right)^3 - 12\left(-\frac{1}{2}\right)^2 + 57\left(-\frac{1}{2}\right) + 18$$
$$= \mathbf{-11\frac{13}{16}}$$

4. (a) Let

$$f(x) = 2x^3 - 3x^2 - 3x + 4$$

Let $x = 1$,

$$\begin{aligned} f(1) &= 2(1)^3 - 3(1)^2 - 3(1) + 4 \\ &= 0 \end{aligned}$$

$$\therefore (x-1) \text{ is a factor of } f(x)$$

Let b be an arbitrary constant,

$$2x^3 - 3x^2 - 3x + 4 = (x-1)\left(2x^2 + bx - 4\right)$$

Comparing the coefficient of x,

$$\begin{aligned} -3 &= -4 - b \\ b &= -1 \end{aligned}$$

$$\therefore f(x) = (x-1)\left(2x^2 - x - 4\right)$$

$$(x-1)\left(2x^2 - x - 4\right) = 0$$

$$x = 1 \quad \text{or} \quad x = \frac{1 \pm \sqrt{33}}{4}$$

$$\therefore x = \mathbf{1} \quad \text{or} \quad x = \mathbf{1.69\ (3.s.f.)} \quad \text{or} \quad x = \mathbf{-1.19\ (3.s.f.)}$$

(b) By the Factor Theorem,

$$\begin{aligned} p(5) + 1 &= 0 \\ p(5) &= -1 \end{aligned}$$

By the Remainder Theorem,

$$\begin{aligned} g(5) &= 2(5)^3 - p(5) + 5 \\ &= \mathbf{256} \end{aligned}$$

5. (a)

$$x^2 - 4 = (x+2)(x-2)$$

$$P(2) = 0$$

$$2(2)^4 + p\left[(2)^3 + (2)^2\right] + q\left[3(2) - 5\right] = 0$$

$$q = -32 - 12p \text{ ......(1)}$$

$$P(-2) = 0$$

$$2(-2)^4 + p\left[(-2)^3 + (-2)^2\right] + q\left[3(-2) - 5\right] = 0$$

$$-4p - 11q = -32 \text{ ......(2)}$$

Substitute Equation (1) into Equation (2),

$$\begin{aligned} -4p - 11(-32 - 12p) &= -32 \\ -4p + 352 + 132p &= -32 \\ p &= -3 \end{aligned}$$

Substitute $p = -3$ into Equation (1),

$$\begin{aligned} q &= -32 - 12(-3) \\ &= 4 \end{aligned}$$

$$p = \mathbf{-3} \qquad q = \mathbf{4}$$

(b)

$$P(x) = 2x^4 - 3x^3 - 3x^2 + 12x - 20$$

$$\begin{aligned} P\left(-\frac{1}{2}\right) &= 2\left(-\frac{1}{2}\right)^4 - 3\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 12\left(-\frac{1}{2}\right) - 20 \\ &= \mathbf{-26\frac{1}{4}} \end{aligned}$$

(c) Let b be an arbitrary constant

$$2x^4 - 3x^3 - 3x^2 + 12x - 20 = \left(x^2 - 4\right)\left(2x^2 + bx + 5\right)$$

Comparing the coefficient of x,

$$\begin{aligned} -4b &= 12 \\ b &= -3 \end{aligned}$$

$$\therefore P(x) = \left(x^2 - 4\right)\left(2x^2 - 3x + 5\right)$$

$$x^2 = 4 \qquad \text{or} \qquad 2x^2 - 3x + 5 = 0$$

For $2x^2 - 3x + 5 = 0$,

$$\begin{aligned} \text{Discriminant} &= (-3)^2 - 4(2)(5) \\ &= -31 < 0 \end{aligned}$$

$$\therefore \text{There are no real roots for } 2x^2 - 3x + 5 = 0$$

$$\therefore \mathbf{2 \text{ solutions}}$$

4 Partial Fractions

4.1 Full Solutions

1. (a) By Long Division,

$$\frac{4x^3+5x^2+x-1}{x^2(x+1)} = 4 + \frac{x^2+x-1}{x^2(x+1)}$$

$$\frac{x^2+x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$
$$x^2+x-1 = Ax(x+1) + B(x+1) + Cx^2$$

Let $x = 0$,

$$(0)^2 + (0) - 1 = A(0)(0+1) + B(0+1) + C(0)^2$$
$$B = -1$$

Let $x = -1$,

$$(-1)^2 + (-1) - 1 = A(-1)(-1+1) + B(-1+1) + C(-1)^2$$
$$C = -1$$

Let $x = 1$,

$$(1)^2 + (1) - 1 = A(1)(1+1) - (1+1) - 1(1)^2$$
$$A = 2$$

$$\therefore \frac{4x^3+5x^2+x-1}{x^2(x+1)} = \mathbf{4 + \frac{2}{x} - \frac{1}{x^2} - \frac{1}{x+1}}$$

(b)

$$\int \frac{4x^3+5x^2+x-1}{x^2(x+1)}\ dx = \int 4 + \frac{2}{x} - \frac{1}{x^2} - \frac{1}{x+1}\ dx$$
$$= \mathbf{4x + 2\ln x + \frac{1}{x} - \ln(x+1) + c}$$

2. (a)

$$\frac{5x^2+4x-3}{x^2(2x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x-1}$$
$$5x^2+4x-3 = Ax(2x-1) + B(2x-1) + Cx^2$$

Let $x = 0$,

$$5(0)^2 + 4(0)^2 - 3 = A(0)(2(0)-1) + B(2(0)-1) + C(0)^2$$
$$B = 3$$

Let $x = \frac{1}{2}$,

$$5\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right) - 3 = A\left(\frac{1}{2}\right)\left[2\left(\frac{1}{2}\right)-1\right] + B\left[2\left(\frac{1}{2}\right)-1\right] + C\left(\frac{1}{2}\right)^2$$
$$C = 1$$

Let $x = 1$,

$$5(1)^2 + 4(1)^2 - 3 = A(1)(2(01-1) + 3(2(1)-1) + (1)^2$$
$$A = 2$$

$$\therefore \frac{5x^2+4x-3}{x^2(2x-1)} = \boldsymbol{\frac{2}{x} + \frac{3}{x^2} + \frac{1}{2x-1}}$$

(b)

$$\begin{aligned}
\int_1^5 \frac{5x^2+4x-3}{x^2(2x-1)}\,dx &= \int_1^5 \frac{2}{x} + \frac{3}{x^2} + \frac{1}{2x-1}\,dx \\
&= \left[2\ln x - \frac{3}{x} + \frac{1}{2}\ln(2x-1)\right]_1^5 \\
&= \left[2\ln 5 - \frac{3}{5} + \frac{1}{2}\ln(2(5)-1)\right] - \left[2\ln 1 - \frac{3}{1} + \frac{1}{2}(2(1)-1)\right] \\
&= 2\ln 5 - \frac{3}{5} + \frac{1}{2}\ln 9 + 3 \\
&= \frac{12}{5} + \ln 25 + \ln 3 \\
&= \frac{12}{5} + \ln 75 \ \textbf{(shown)}
\end{aligned}$$

□

3. (a) Let

$$f(x) = 2x^3 - 13x^2 + 24x - 9$$

Let $x = 3$,

$$\begin{aligned} f(3) &= 2(3)^3 - 13(3)^2 + 24(3) - 9 \\ &= 0 \end{aligned}$$

$$\therefore (x-3) \text{ is a factor of } f(x)$$

(b) Let b be an arbitrary constant

$$2x^3 - 13x^2 + 24x - 9 = (x-3)\left(2x^2 + bx + 3\right)$$

Comparing the coefficient of x,

$$\begin{aligned} 24 &= -3b + 3 \\ b &= -7 \end{aligned}$$

$$\begin{aligned} f(x) &= (x-3)\left(2x^2 - 7x + 3\right) \\ &= (x-3)(2x-1)(x-3) \\ &= (2x-1)(x-3)^2 \end{aligned}$$

$$\begin{aligned} \frac{5x^2 - 30x + 10}{(2x-1)(x-3)^2} &= \frac{A}{2x-1} + \frac{B}{x-3} + \frac{C}{(x-3)^2} \\ 5x^2 - 30x + 10 &= A(x-3)^2 + B(x-3)(2x-1) + C(2x-1) \end{aligned}$$

Let $x = 3$,

$$\begin{aligned} 5(3)^2 - 30(3) + 10 &= A(3-3)^2 + B(3-3)(2(3)-1) + C(2(3)-1) \\ C &= -7 \end{aligned}$$

Let $x = \frac{1}{2}$,

$$\begin{aligned} 5\left(\frac{1}{2}\right)^2 - 30\left(\frac{1}{2}\right) + 10 &= A\left[\left(\frac{1}{2}\right) - 3\right]^2 + B\left[\left(\frac{1}{2}\right) - 3\right] \cdot \left[2\left(\frac{1}{2}\right) - 1\right] + C\left[2\left(\frac{1}{2}\right) - 1\right] \\ A &= -\frac{3}{5} \end{aligned}$$

Let $x = 0$,

$$\begin{aligned} 5(0)^2 - 30(0) + 10 &= -\frac{3}{5}(0-3)^2 + B(0-3)(2(0)-1) - 7(2(0)-1) \\ B &= \frac{14}{5} \end{aligned}$$

$$\therefore \frac{5x^2 - 30x + 10}{(2x-1)(x-3)^2} = \mathbf{-\frac{3}{5(2x-1)} + \frac{14}{5(x-3)} - \frac{7}{(x-3)^2}}$$

(c)

$$\begin{aligned} \int \frac{10x^2 - 60x + 20}{2x^3 - 13x^2 + 24x - 9}\, dx &= 2\int \frac{5x^2 - 30x + 10}{(2x-1)(x-3)^2}\, dx \\ &= 2\int -\frac{3}{5(2x-1)} + \frac{14}{5(x-3)} - \frac{7}{(x-3)^2} +\, dx \\ &= 2\left[-\frac{3}{5(2)}\ln(2x-1) + \frac{14}{5}\ln(x-3) - \frac{7}{(-1)(x-3)} + c\right] \\ &= \mathbf{-\frac{3}{5}\ln(2x-1) + \frac{28}{5}\ln(x-3) + \frac{14}{(x-3)} + c} \end{aligned}$$

4. (a)

$$\begin{aligned} x^3 + 8 &= x^3 + 2^2 \\ &= \mathbf{(x+2)\left(x^2 - 2x + 4\right)} \end{aligned}$$

(b) (i)

$$\begin{aligned} \text{Volume} &= \frac{1}{3}\,(\text{Base Area})\,(\text{Height}) \\ \frac{1}{3}\left(x^3+8\right)(\text{Height}) &= x^3 + \frac{1}{3}x^2 + \frac{14}{3}x + 4 \\ \text{Height} &= \frac{3x^3 + x^2 + 14x + 12}{x^3+8} \end{aligned}$$

By **long division**,

$$h = \mathbf{3 + \frac{x^2 + 14x - 12}{x^3 + 8}}$$

(ii)

$$\begin{aligned} \frac{x^2 + 14x - 12}{(x+2)\left(x^2 - 2x + 4\right)} &= \frac{D}{x+2} + \frac{Ex + G}{x^2 - 2x + 4} \\ x^2 + 14x - 12 &= D\left(x^2 - 2x + 4\right) + (Ex + G)(x + 2) \end{aligned}$$

Let $x = -2$,

$$\begin{aligned} (-2)^2 + 14(-2) - 12 &= D\left[(-2)^2 - (-2)^2 + 4\right] + [E(-2) + G]\cdot[(-2) + 2] \\ -36 &= 12D \\ D &= -3 \end{aligned}$$

Let $x = 0$,

$$\begin{aligned} (0)^2 + 14(0) - 12 &= (-3)\left[(0)^2 - (0)^2 + 4\right] + [E(0) + G]\cdot[(0) + 2] \\ -12 &= 4(-3) + 2G \\ G &= 0 \end{aligned}$$

Let $x = 1$,

$$\begin{aligned} (1)^2 + 14(1) - 12 &= (-3)\left[(1)^2 - (1)^2 + 4\right] + [E(1) + 0]\cdot[(1) + 2] \\ 3 &= -9 + 3E \\ E &= 4 \end{aligned}$$

$$\therefore h = \mathbf{3 - \frac{3}{x-2} + \frac{4x}{x^2 - 2x + 4}}$$

5 Binomial Theorem

5.1 Full Solutions

1. (a)

$$\begin{aligned}\left(x^5+\frac{2}{x^6}\right)^n &= \left(x^5\right)^n+\binom{n}{1}\left(x^5\right)^{n-1}\left(\frac{2}{x^6}\right)+\binom{n}{2}\left(x^5\right)^{n-2}\left(\frac{2}{x^6}\right)^2+... \\ &= x^{5n}+n\left(x^{5n-5}\right)\left(2x^{-6}\right)+\frac{n(n-1)}{2}\left(x^{5n-10}\right)\left(4x^{-12}\right) \\ &= \boldsymbol{x^{5n}+2nx^{5n-11}+2n(n-1)x^{5n-22}+...}\end{aligned}$$

(b)

$$\begin{aligned}\frac{2n(n-1)}{2n} &= 8 \\ n &= 9 \ \textbf{(shown)}\end{aligned}$$

□

(c)

$$\begin{aligned}T_{n+1} &= \binom{9}{r}\left(x^5\right)^{9-r}\left(\frac{2}{x^6}\right)^r \\ &= \binom{9}{r}(2)^r\left(x^{45-11r}\right)\end{aligned}$$

For the constant term, x^0

$$\begin{aligned}45-11r &= 0 \\ r &= \frac{45}{11}\notin\mathbb{Z}^+ \quad \Rightarrow\Leftarrow\end{aligned}$$

$\therefore$ There is no constant term **(shown)**

□

2. (a)

$$T_{r+1} = \binom{8}{r}\left(\frac{a^2}{\sqrt{x}}\right)^{8-r}\left(-\frac{\sqrt{x}}{a}\right)^{r}$$
$$= \binom{8}{r}(-1)^r a^{16-3r}x^{r-4}$$

For the independent term, x^0

$$r-4=0$$
$$r=4$$

$$\text{Term independent of } x = \binom{8}{4} a^{16-3(4)}(-1)^4$$
$$= \mathbf{70a^4}$$

(b)

$$\left(\frac{3x^4-4x^2}{x^2}\right)\left(\frac{a^2}{\sqrt{x}}-\frac{\sqrt{x}}{a}\right)^8 = \left(3x^2-4\right)\left(... + x^2 \text{ term} + \text{independent term} + ...\right)$$

For the x^2 term,

$$r-4=2$$
$$r=6$$

$$\text{Term in } x^2 = \binom{8}{6} a^{16-3(6)}x^{6-4}(-1)^6$$
$$= \frac{28}{a^2}x^2$$

$$\left(\frac{3x^4-4x^2}{x^2}\right)\left(\frac{a^2}{\sqrt{x}}-\frac{\sqrt{x}}{a}\right)^8 = \left(3x^2-4\right)\left(... + \frac{28}{a^2}x^2 + 70a^2 + ...\right)$$
$$= ... + 210a^4x^2 - \frac{112}{a^2}x^2 + ...$$

$$\therefore \text{Coefficient of } x^2 = \mathbf{210a^4 - \frac{112}{a^2}}$$

3. (a)

$$(1+x)^7 = 1^7 + \binom{7}{1}(1)^{7-1}x + \binom{7}{2}(1)^{7-2}(x)^2 + \binom{7}{3}(1)^{7-3}(x)^3 + ...$$
$$= \mathbf{1 + 7x + 21x^2 + 35x^3 + ...}$$

(b)

$$T_{r+1} = \binom{9}{r}\left(x^2\right)^{9-r}\left(-\frac{2}{x^3}\right)^r$$
$$= \boldsymbol{\binom{9}{r}(-2)^r x^{18-5r}}$$

(c)

$$\text{Power} = \mathbf{18 - 5r}$$

(d) For the x^3 term,

$$18 - 5r = 3$$
$$r = 3$$

$$\text{Coefficient of } x^3 = 35 + \binom{9}{3}(-2)^3$$
$$= \mathbf{-637}$$

4. (a)

$$(3-px)^5 + (2+x)^6 = \left[... + \binom{5}{3}(3)^{5-3}(-px)^3 + ...\right] + \left[... + \binom{6}{3}(2)^{6-3}(x)^3 + ...\right]$$
$$= ...\left(-90p^3 + 160\right)x^3 + ...$$

$$\therefore -90p^3 + 160 = \frac{595}{4}$$
$$p^3 = \frac{1}{8}$$
$$p = \mathbf{\frac{1}{2}}$$

(b)

$$\left(x^2 - 2x\right)^2(2+x)^6 = \left(x^4 - 4x^3 + 4x^2\right)\left(2^6 + \binom{6}{1}(2)^{6-1}(x) + ...\right)$$
$$= ... + 512x^3 + ...$$

$$\therefore \text{Coefficient of } x^3 = \mathbf{512}$$

6 Exponential & Logarithms

6.1 Full Solutions

1. (a) When $t = 0$,

$$\begin{aligned} N &= 8000\left(2 + 3e^{-\frac{0}{50}}\right) \\ &= 8000\left(2 + 3e^{0}\right) \\ &= \mathbf{40\ 000} \end{aligned}$$

(b) When $t = 50$,

$$\begin{aligned} N &= 8000\left(2 + 3e^{-\frac{50}{50}}\right) \\ &= 8000\left(2 + 3e^{-1}\right) \\ &= 24829.10... \\ &= \mathbf{24800\ (3.s.f.)} \end{aligned}$$

(c)

$$\begin{aligned} 20\ 000 &= 8000\left(2 + 3e^{-\frac{t}{50}}\right) \\ e^{-\frac{t}{50}} &= \frac{1}{6} \\ -\frac{t}{50} &= \ln\frac{1}{6} \\ t &= -50\ln\frac{1}{6} \\ &= 89.587973... \\ &\approx \mathbf{90\ years} \end{aligned}$$

(d)

$$\begin{aligned} \frac{dN}{dt} &= 24000\left(-\frac{t}{50}\right)e^{-\frac{t}{50}} \\ &= -480e^{-\frac{t}{50}} \end{aligned}$$

$$\begin{aligned} \left.\frac{dN}{dt}\right|_{t=10} &= -480e^{-\frac{10}{50}} \\ &= -392.990761... \\ &= -393 \end{aligned}$$

$\therefore$ The rate is decreasing at a rate of **393** polar bears/year

(e)

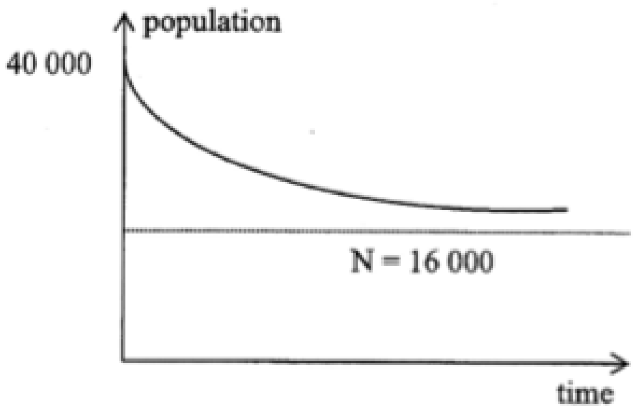
$$t \to \infty \quad \Rightarrow \quad e^{-\frac{t}{50}} \to 0$$

$$\begin{aligned} N &\to 8000(2) \\ &= 16000 \end{aligned}$$

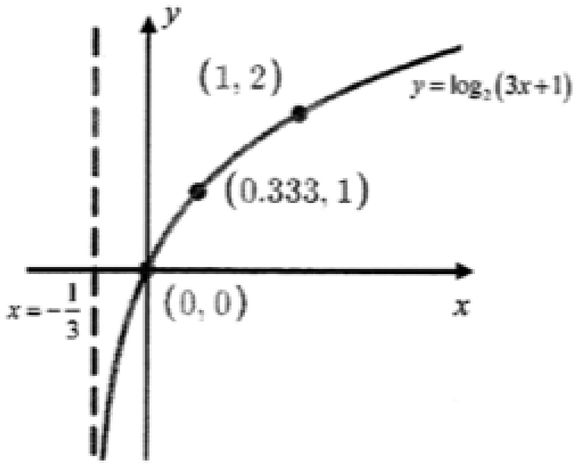
$\therefore$ The population will never fall below 16 000

□

(f) Diagram



2. (a) (i) Diagram



(ii)

$$y = \log_2(3x+1)$$
$$2^y = 3x+1$$

Since $2^y > 0$,

$$3x + 1 > 0$$
$$x > -\frac{1}{3} \textbf{ (shown)}$$

□

(b)

$$\log_2(3x+1) + \frac{1}{2}\log_{\sqrt{2}}(3x-1) = 1$$
$$\log_2(3x+1) + \frac{1}{2}\left[\frac{\log_2(3x-1)}{\log_2\sqrt{2}}\right] = 1$$
$$\log_2(3x+1) + \log_2(3x-1) = 1$$
$$\log_2\left[(3x+1)(3x-1)\right] = 1$$
$$\therefore 9x^2 - 1 = 2$$
$$x^2 = \frac{1}{3}$$

$$x = \mathbf{\frac{1}{\sqrt{3}}} \qquad \text{or} \qquad x = -\frac{1}{\sqrt{3}} \text{ (rej)}$$

3. (a) (i)

$$\begin{aligned}\log_2 1 - p + q &= 0 - 2^x + 2^y \\ &= \mathbf{2^y - 2^x}\end{aligned}$$

(ii)

$$\begin{aligned}\log_2 \sqrt{\frac{p^5}{q^3}} &= \frac{1}{2}\left[\log_2 p^5 - \log_2 q^3\right] \\ &= \mathbf{\frac{1}{2}\left(5x - 3y\right)}\end{aligned}$$

(iii)

$$\begin{aligned}\log_{\sqrt{2}} 4p &= \frac{\log_2 4 + \log_2 p}{\log_2 \sqrt{2}} \\ &= \mathbf{2(2 + x)}\end{aligned}$$

(b)

$$\begin{aligned}4\log_4 x + 1 &= 3\log_8(5 - 3x) \\ 4\left(\frac{\log_2 x}{\log_2 4}\right) + 1 &= 3\left(\frac{\log_2(5 - 3x)}{\log_2 8}\right) \\ \log_2(5 - 3x) - 2\log_2 x &= 1 \\ \log_2\left(\frac{5 - 3x}{x^2}\right) &= 1 \\ \frac{5 - 3x}{x^2} &= 2 \\ 2x^2 + 3x - 5 &= 0 \\ (x - 1)(2x + 5) &= 0\end{aligned}$$

$$\therefore x = \mathbf{1} \qquad \text{or} \qquad x = -\frac{5}{2} \text{ (rej)}$$

4. (a)

$$\begin{aligned}
2\log_5 x + \log_{25} 16 &= \log_5(9x-2) \\
2\log_5 x + \frac{\log_5 16}{\log_5 25} &= \log_5(9x-2) \\
2\log_5 x + \frac{1}{2}\log_5 16 &= \log_5(9x-2) \\
\log_5 x^2 + \log_5 4 &= \log_5(9x-2)
\end{aligned}$$

$$\therefore 4x^2 = 9x - 2$$

$$4x^2 - 9x + 2 = 0$$

$$(4x-1)(x-2) = 0$$

$$\therefore x = \mathbf{\frac{1}{4}} \qquad \text{or} \qquad x = \mathbf{2}$$

(b)

$$\begin{aligned}
\frac{1}{\log_{ab} a} - \frac{1}{\log_{ab} b} &= \log_a ab - \log_b ab \\
&= \log_a a + \log_a b - \log_b a - \log_b b \\
&= \log_a b - \log_b a \\
&= \frac{1}{\log_b a} - \frac{1}{\log_a b} \\
&= \boldsymbol{-\sqrt{293}}
\end{aligned}$$

5. (a) (i) When $I = I_0$,

$$M = \lg\left(\frac{I_0}{I_0}\right)$$
$$= \mathbf{0}$$

(ii)

$$5.8 = \lg\left(\frac{I_T}{I_0}\right)$$
$$5.8 = \lg I_T - \lg I_0 \text{ ......(1)}$$

$$6.3 = \lg\left(\frac{l_C}{l_0}\right)$$
$$6.3 = \lg I_C - \lg I_0 \text{ ......(2)}$$

Taking Equation (2) - Equation (1),

$$0.5 = \lg I_C - \lg I_T$$
$$= \lg\left(\frac{l_C}{l_T}\right)$$
$$\therefore \frac{l_C}{l_T} = \mathbf{10^{0.5}}$$

(b)

$$2^{p-9} \div 8^q - \sqrt[4]{32^p} \text{ ......(1)}$$
$$\log_2 6 - \log_4(11q - 2p) = 1 \text{ ......(2)}$$

From Equation (1),

$$2^{p-9} \div 2^{3q} = \left(2^{5p}\right)^{\frac{1}{4}}$$
$$2^{p-9-3q} = 2^{\frac{5p}{4}}$$
$$\therefore p - 9 - 3q = \frac{5p}{4}$$
$$p = -12q - 36 \text{ ......(3)}$$

From Equation (2),

$$\log_2 6 - \frac{\log_2(11q - 2p)}{\log_2 4} = 1$$
$$2\log_2 6 - \log_2(11q - 2p) = 2$$
$$\log_2 \frac{36}{11q - 2p} = 2$$
$$\therefore \frac{36}{11q - 2p} = 2^2$$
$$11q - 2p = 9 \text{ ......(4)}$$

Substitute Equation (3) into Equation (4),

$$11q - 2(-12q - 36) = 9$$
$$q = -\mathbf{\frac{9}{5}}$$

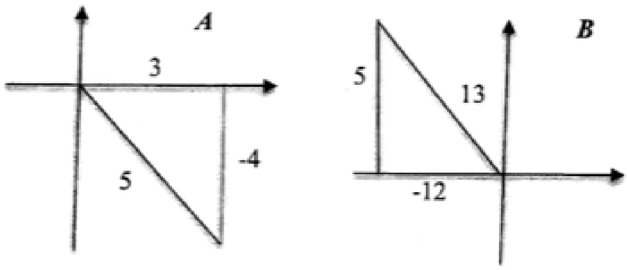
Substitute $q = -\frac{9}{5}$ into Equation (3),

$$p = -12\left(-\frac{9}{5}\right) - 36$$
$$= \mathbf{-\frac{72}{5}}$$

7 Trigonometry

7.1 Full Solutions

1. First, note that A is in the 4th quadrant, B is in the 2nd quadrant



(a)

$$\begin{aligned}\cot A &= \frac{1}{\tan A} \\ &= \frac{1}{\left(-\frac{4}{3}\right)} \\ &= -\mathbf{\frac{3}{4}}\end{aligned}$$

(b)

$$\begin{aligned}\cos(A+B) &= \cos A \cos B - \sin A \sin B \\ &= \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) \\ &= -\mathbf{\frac{16}{65}}\end{aligned}$$

(c)

$$\begin{aligned}\sin\left(\frac{B}{2}\right) &= \sqrt{\frac{1-\cos B}{2}} \text{ (rej -ve)} \\ &= \sqrt{\frac{1-\left(-\frac{12}{13}\right)}{2}} \\ &= \sqrt{\frac{25}{26}} \\ &= \frac{5}{\sqrt{26}} \\ &= \mathbf{\frac{5\sqrt{26}}{26}}\end{aligned}$$

2. (a)

$$TX = 16\cos\theta \qquad XU = 16\sin\theta \qquad WU = 6\cos\theta \qquad WV = 6\sin\theta$$

$$\begin{aligned} P &= 16 + 6 + 6\sin\theta + (16\sin\theta - 6\cos\theta) + 16\cos\theta \\ &= 22 + 10\cos\theta + 22\sin\theta \ \textbf{(shown)} \end{aligned}$$

□

(b)

$$\begin{aligned} R &= \sqrt{(10)^2 + (22)^2} \\ &= \sqrt{584} \\ &= 2\sqrt{146} \end{aligned}$$

$$\begin{aligned} \tan\alpha &= \frac{10}{22} \\ \alpha &= \tan^{-1}\left(\frac{10}{22}\right) \\ &= 24.443954... \\ &= 24.4^\circ \ \textbf{(1.d.p.)} \end{aligned}$$

$$\therefore P = \mathbf{22 + 2\sqrt{146}\sin(\boldsymbol{\theta} + 24.4^\circ)}$$

(c)

$$\begin{aligned} P_{\text{max}} &= 22 + 2\sqrt{146} \\ &= 46.16609...\text{ cm} < 45\text{ cm} \end{aligned}$$

$\therefore$ Hence, it is possible for P to be 45 cm

□

(d) When $P = 45$,

$$\begin{aligned} 22 + 2\sqrt{146}\sin\left[\theta + \tan^{-1}\left(\frac{10}{22}\right)\right] &= 45 \\ \sin\left[\theta + \tan^{-1}\left(\frac{10}{22}\right)\right] &= \frac{23}{\sqrt{584}} \end{aligned}$$

$$\alpha = \sin^{-1}\left(\frac{23}{\sqrt{584}}\right) \qquad \text{(Quadrant 1 or 2)}$$

For Quadrant 1,

$$\begin{aligned} \theta &= \sin^{-1}\left(\frac{23}{\sqrt{584}}\right) - \tan^{-1}\left(\frac{10}{22}\right) \\ &= 47.684470... \\ &= \mathbf{47.7^\circ} \ \textbf{(1.d.p.)} \end{aligned}$$

For Quadrant 2,

$$\begin{aligned} \theta &= \pi - \sin^{-1}\left(\frac{23}{\sqrt{584}}\right) - \tan^{-1}\left(\frac{10}{22}\right) \\ &= 83.427619... \\ &= \mathbf{83.4^\circ} \ \textbf{(1.d.p.)} \end{aligned}$$

3. (a)

$$\begin{aligned}
\text{LHS} &= \cos(A+B)\cos(A-B) \\
&= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B) \\
&= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B \\
&= \cos^2 A\cos^2 B - \left(1-\cos^2 A\right)\left(1-\cos^2 B\right) \\
&= \cos^2 A\cos^2 B - \left[1-\cos^2 A-\cos^2 B+\cos^2 A\cos^2 B\right] \\
&= \cos^2 A+\cos^2 B-1 \\
&= \text{RHS } \textbf{(shown)}
\end{aligned}$$

□

(b)

$$\begin{aligned}
\cos 15^\circ \cos 75^\circ &= \cos(45^\circ-30^\circ)\cos(45^\circ+30^\circ) \\
&= (\cos 45^\circ)^2 + (\cos 30^\circ)^2 - 1 \\
&= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 - 1 \\
&= \frac{1}{2}+\frac{3}{4}-1 \\
&= \mathbf{\frac{1}{4}}
\end{aligned}$$

4. (a) (i)

$$\begin{aligned}
\text{LHS} &= \sin x \cos x + \cot x \cos^2 x \\
&= \cos x \left(\sin x + \cot x \cos x\right) \\
&= \cos x \left[\sin x + \left(\frac{\cos x}{\sin x}\right) \cos x\right] \\
&= \cos x \left(\sin x + \frac{\cos^2 x}{\sin x}\right) \\
&= \cos x \left(\frac{\sin^2 x + \cos^2 x}{\sin x}\right) \\
&= \cos x \left(\frac{1}{\sin x}\right) \\
&= \cot x \\
&= \text{RHS } \textbf{(shown)}
\end{aligned}$$

□

(ii) From part (a)(i),

$$\begin{aligned}
\cot 3x &= 1 \\
\tan 3x &= 1
\end{aligned}$$

$$\begin{aligned}
\alpha &= \tan^{-1}(1) \\
&= \frac{\pi}{4} \quad \text{(Quadrant 1 or 3)}
\end{aligned}$$

For Quadrant 1 (1st rotation),

$$\begin{aligned}
3x &= \frac{\pi}{4} \\
x &= \boldsymbol{\frac{\pi}{12}}
\end{aligned}$$

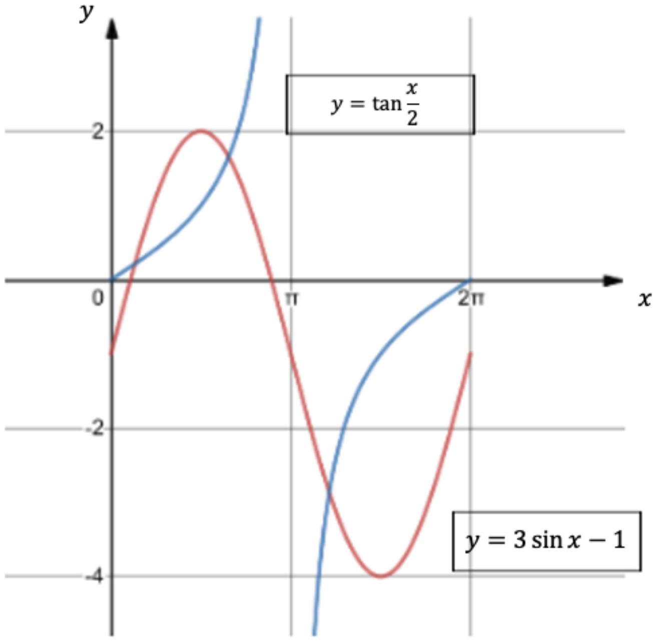
For Quadrant 3,

$$\begin{aligned}
3x &= \pi + \frac{\pi}{4} \\
&= \frac{5\pi}{4} \\
\therefore x &= \boldsymbol{\frac{5\pi}{12}}
\end{aligned}$$

For Quadrant 1 (2nd rotation),

$$\begin{aligned}
3x &= 2\pi + \frac{\pi}{4} \\
&= \frac{9\pi}{4} \\
\therefore x &= \boldsymbol{\frac{3\pi}{4}}
\end{aligned}$$

(b) (i) Diagram



(ii)

3 solutions

5. (a) Note that A and C are the maximum and minimum points of the curve, which is the amplitude

$$\therefore 2 \times 3 = 6 \text{ cm } \textbf{(shown)}$$

$\square$

(b)

$$\begin{aligned}\text{Period} &= 2 \times 0.25 \\ &= 0.5 \text{ seconds}\end{aligned}$$

$$\begin{aligned}\therefore b &= \frac{2\pi}{0.5} \\ &= 4\pi \text{ rad/s}\end{aligned}$$

$$\therefore k = 4 \textbf{ (shown)}$$

$\square$

(c)

$$\begin{aligned}-3\cos(4\pi t) + 7 &= 8 \\ \cos(4\pi t) &= -\frac{1}{3}\end{aligned}$$

$$\alpha = \cos^{-1}\left(\frac{1}{3}\right) \qquad \text{(Quadrant 2 or 3)}$$

For Quadrant 2,

$$\begin{aligned}t &= \frac{\pi - \cos^{-1}\left(\frac{1}{3}\right)}{4\pi} \\ &= 0.152043... \\ &= \textbf{0.152 s (3.s.f.)}\end{aligned}$$

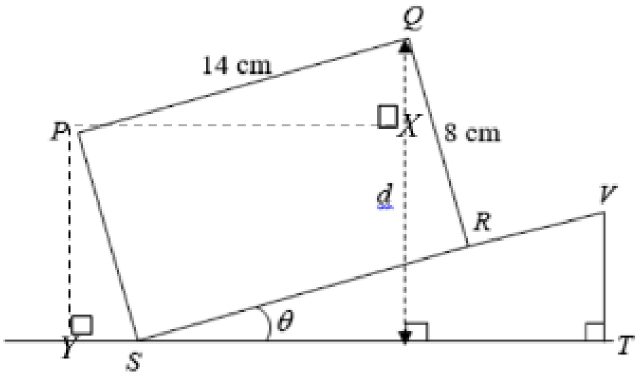
For Quadrant 2,

$$\begin{aligned}t &= \frac{\pi + \cos^{-1}\left(\frac{1}{3}\right)}{4\pi} \\ &= 0.347956... \\ &= \textbf{0.348 s (3.s.f.)}\end{aligned}$$

(d)

$$\begin{aligned}\text{Duration} &= \frac{\left(\frac{\pi + \cos^{-1}\left(\frac{1}{3}\right)}{4\pi}\right) - \left(\frac{\pi - \cos^{-1}\left(\frac{1}{3}\right)}{4\pi}\right)}{2} \\ &= 0.0979566... \\ &= \textbf{0.0980 s (3.s.f.)}\end{aligned}$$

6. (a) From this diagram:



$$\angle QPX = \theta \text{ (corresponding angles)}$$
$$\therefore QX = 14\sin\theta$$

$$\begin{aligned}\angle PSY &= 180^\circ - 90^\circ - \theta \\ &= 90^\circ - \theta \text{ (adjacent angles on a straight line)}\end{aligned}$$

$$\begin{aligned}\sin\angle PSY &= \frac{PY}{8} \\ \therefore PY &= 8\sin(90^\circ - \theta) \\ &= 8\cos\theta\end{aligned}$$

$$\begin{aligned}\therefore d &= PY + QX \\ &= 8\cos\theta + 14\sin\theta \ \textbf{(shown)}\end{aligned}$$

□

(b)

$$\begin{aligned}R &= \sqrt{(8)^2 + (14)^2} \\ &= \sqrt{260}\end{aligned}$$

$$\begin{aligned}\alpha &= \tan^{-1}\left(\frac{8}{14}\right) \\ &= 29.7^\circ \ \text{(1.d.p.)}\end{aligned}$$

$$\therefore d = \boldsymbol{\sqrt{260}\sin(\theta + 29.7^\circ)}$$

(c)

$$\sqrt{260}\sin\left[\theta+\tan^{-1}\left(\frac{8}{14}\right)\right]=\sqrt{200}$$
$$\sin\left[\theta+\tan^{-1}\left(\frac{8}{14}\right)\right]=\sqrt{\frac{200}{260}}$$

$$\alpha=\sin^{-1}\left(\sqrt{\frac{10}{13}}\right)\qquad\text{(Quadrant 1)}$$

For Quadrant 1,

$$\begin{aligned}\theta&=\sin^{-1}\left(\sqrt{\frac{10}{13}}\right)-\tan^{-1}\left(\frac{8}{14}\right)\\&=31.544603...\\&=\mathbf{31.5^\circ\ (1.d.p.)}\end{aligned}$$

(d)

$$d_{\max}=\mathbf{2\sqrt{65}}$$

8 Coordinate Geometry

8.1 Full Solutions

1. (a) Since ABC is a right-angled triangle

$$\begin{aligned} m_{AB} \times m_{AC} &= -1 \\ \left(\frac{0-8}{k-2}\right) \times \left(\frac{0-(-4)}{k-(-2)}\right) &= -1 \\ -36 &= -(k-2)(k+2) \\ -32 &= -k^2+4 \\ k^2 &= 36 \\ k &= \pm 6 \text{ (rej -ve)} \\ \therefore k &= \mathbf{6} \end{aligned}$$

(b) Let the coordinates of N be $(0, n)$

$$\begin{aligned} m_{BN} &= m_{BC} \\ \frac{8-n}{2-0} &= \frac{8-(-4)}{2-(-2)} \\ \frac{8-n}{2} &= 3 \\ n &= 2 \\ \therefore N&(0,2) \end{aligned}$$

$$\begin{aligned} \text{Mid-point of } BC &= \left(\frac{2-2}{2}, \frac{8-4}{2}\right) \\ &= (0,2) \\ &= \text{Coordinates of } N \ \textbf{(shown)} \end{aligned}$$

□

(c)

$$\begin{aligned} \text{Gradient of } AC &= \frac{0-(-4)}{6-(-2)} \\ &= \frac{1}{2} \end{aligned}$$

Hence, the equation of AC is

$$\begin{aligned} y - 0 &= \frac{1}{2}(x-6) \\ y &= \frac{1}{2}x - 3 \end{aligned}$$

Let the coordinates of M be $\left(a, \frac{1}{2}a - 3\right)$

$$\begin{aligned} \text{Area of quadrilateral } ABNM &= 25 \text{ units}^2 \\ \frac{1}{2}\begin{vmatrix} 6 & 2 & 0 & a & 6 \\ 0 & 8 & 2 & \frac{1}{2}a - 3 & 0 \end{vmatrix} &= 25 \\ \frac{1}{2}\left[(48+4) - \left(-2a + 6\left(\frac{1}{2}a - 3\right)\right)\right] &= 25 \\ 5a &= 20 \\ a &= 4 \\ \therefore M&\mathbf{(4, -1)} \end{aligned}$$

2. (a)

$$\begin{aligned}\frac{3-k}{2-(-2)} &= 1.5\\ 2-k &= 6\\ k &= \mathbf{-3}\end{aligned}$$

(b)

$$\text{Gradient of } BD = -\frac{2}{3}$$

Hence, the equation of BD is

$$y-(-2) = -\frac{2}{3}(x-1)$$

$$\therefore \boldsymbol{y = -\frac{2}{3}x - \frac{4}{3}}$$

(c)

$$y = \frac{3}{2}x \text{ ......(1)}$$

$$y = -\frac{2}{3}x - \frac{4}{3} \text{ ......(2)}$$

Let Equation (1) = Equation (2),

$$\begin{aligned}\frac{3}{2}x &= -\frac{2}{3}x - \frac{4}{3}\\ \frac{13}{6}x &= -\frac{4}{3}\\ x &= -\frac{8}{13}\end{aligned}$$

Substitute $x = -\frac{8}{13}$ into Equation (1),

$$\begin{aligned}y &= \frac{3}{2}\left(-\frac{8}{13}\right)\\ &= -\frac{12}{13}\end{aligned}$$

$$\therefore \boldsymbol{M\left(-\frac{8}{13}, -\frac{12}{13}\right)}$$

(d) Since $ABCD$ is a kite,

$$\begin{aligned}DM &= BM\\ \sqrt{\left(a+\frac{8}{13}\right)^2 + \left(b+\frac{12}{13}\right)^2} &= \sqrt{\left(-\frac{8}{13}-1\right)^2 + \left(-\frac{12}{13}+2\right)^2}\\ \sqrt{\left(a+\frac{8}{13}\right)^2 + \left(b+\frac{12}{13}\right)^2} &= \sqrt{\frac{49}{13}}\\ \left(a+\frac{8}{13}\right)^2 + \left(b+\frac{12}{13}\right)^2 &= \frac{49}{13}\end{aligned}$$

$$13\left(a+\frac{8}{13}\right)^2 + 13\left(b+\frac{12}{13}\right)^2 = 49 \textbf{ (shown)}$$

□

3. (a)

$$\begin{aligned}\sqrt{(3a-2)^2+(2a+4-0)^2} &= 4\sqrt{5}\\ \sqrt{9a^2-12a+4+4a^2+16a+16} &= \sqrt{80}\\ 13a^2+4a+20 &= 80\\ 13a^2+4a-60 &= 0\\ (a-2)(13a+30) &= 0\end{aligned}$$

$$a = \mathbf{2} \qquad \text{or} \qquad a = -\frac{30}{13} \text{ (rej)}$$

(b)

$$\begin{aligned}\text{Gradient of } AD &= \frac{2-0}{-2-2}\\ &= -\frac{1}{2}\end{aligned}$$

$$\therefore \text{Gradient of } DC = 2$$

Hence, the equation of CD is

$$\begin{aligned}y-2 &= 2(x+2)\\ y &= 2x+6\end{aligned}$$

$$\therefore \boldsymbol{C(0,6)}$$

(c)

$$\begin{aligned}\text{Midpoint of } AB &= \left(\frac{6+2}{2}, \frac{8+0}{2}\right)\\ &= (4,4)\end{aligned}$$

Hence, the equation of the perpendicular bisector is

$$\begin{aligned}y-4 &= -\frac{1}{2}(x-4)\\ \boldsymbol{y} &= \boldsymbol{-\frac{1}{2}x+6}\end{aligned}$$

(d) Yes, the point $C(0,6)$ lies on the perpendicular bisector as the y-intercept of the perpendicular bisector has a coordinate of $(0,6)$

(e)

$$\begin{aligned}\text{Area of trapezium } ABCD &= \frac{1}{2}\begin{vmatrix}0 & -2 & 2 & 6 & 0\\ 6 & 2 & 0 & 8 & 6\end{vmatrix}\\ &= \frac{1}{2}\left|52-(-8)\right|\\ &= \frac{1}{2}\left|60\right|\\ &= \mathbf{30\ units^2}\end{aligned}$$

4. (a)

$$y = x - \frac{1}{2}$$

Substitute $y = 0$,

$$0 = x - \frac{1}{2}$$
$$x = \frac{1}{2}$$
$$\therefore D\left(\frac{1}{2}, 0\right)$$

' Let the coordinates of C be (x_c, y_c)

$$\text{Midpoint of } AC = \text{Midpoint of } BD$$
$$\left(\frac{-0.5 + x_c}{2}, \frac{2 + y_c}{2}\right) = \left(\frac{1 + 0.5}{2}, \frac{3.5 + 0}{2}\right)$$
$$\therefore x_c = 2 \qquad y_c = 1.5$$
$$\therefore \boldsymbol{C\left(2, 1\frac{1}{2}\right)}$$

By inspection, using $3BE = BC$

$$\boldsymbol{E\left(1\frac{1}{3}, 2\frac{5}{6}\right)}$$

(b) At F, substitute $x = \frac{4}{3}$ into CD,

$$y = \frac{4}{3} - \frac{1}{2}$$
$$= \frac{5}{6}$$
$$\therefore \boldsymbol{F\left(1\frac{1}{3}, \frac{5}{6}\right)}$$

(c) First, note that AN and EF are parallel, and are parallel to the y-axis

$$\text{Gradient of } AE = \frac{\frac{17}{6} - 2}{\left(\frac{4}{3}\right) - \left(-\frac{1}{2}\right)}$$
$$= \frac{5}{11}$$

$$\text{Gradient of } NF = \frac{\frac{5}{6} - 0}{\left(\frac{4}{3}\right) - \left(-\frac{1}{2}\right)}$$
$$= \frac{5}{11}$$

$$\therefore \text{Gradient of } AE = \text{Gradient of } NF$$

Since $AEFN$ is a quadrilateral with 2 pairs of parallel sides, it is a parallelogram **(shown)**

□

9 Further Coordinate Geometry

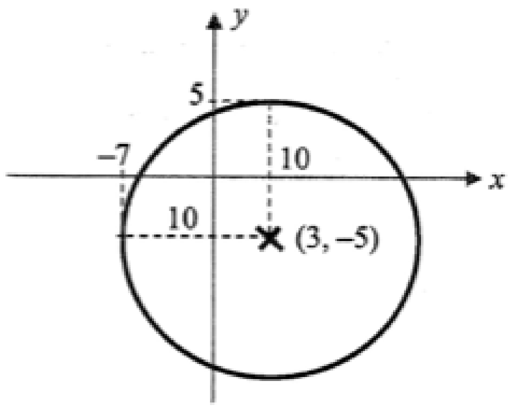
9.1 Full Solutions

1. (a)

$$\begin{aligned} x^2 - 6x + y^2 + 10y &= 66 \\ (x-3)^2 - 9 + (y+5)^2 - 25 &= 66 \\ (x-3)^2 + (y+5)^2 &= 66 + 9 + 25 \\ (x-3)^2 + (y+5)^2 &= 10^2 \end{aligned}$$

$$\therefore \text{Centre of } C_1 = (3, -5)$$

$$\therefore \text{Radius of } C_1 = 10 \text{ units}$$



The centre of the circle is 5 units from the x-axis and 3 units from the$y$-axis. As the radius of the circle (10 units) is larger than the distance of the centre from both axes ($10 > 3$ and $10 > 5$), the circle will intersect both axes twice. Hence, they are **not tangents** to the circle C_1

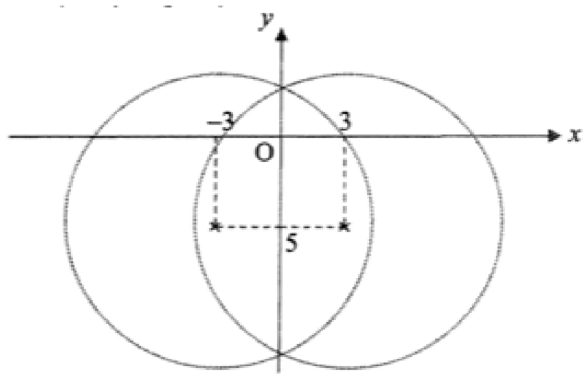
(b)

$$\begin{aligned} \text{Distance} &= \sqrt{(3-2)^2 + (-5-(-4))^2} \\ &= \sqrt{2} < 10 \end{aligned}$$

Since the distance between $(2, -4)$ and the centre is smaller than the radius, therefore the point $(2, -4)$ lies **inside** the circle

(c)

$$\text{New centre of } C_2 = (-3, -5)$$



$$\therefore \boldsymbol{(x+3)^2 + (y+5)^2 = 100}$$

2. (a)

$$x^2 + y^2 + px + \left(\frac{p}{2} + 4\right) y + k = 0$$
$$\therefore C\left(-\frac{p}{2}, -\frac{p}{4} - 2\right)$$

Substitute C into the line,

$$3\left(-\frac{p}{2}\right) - 2\left(-\frac{p}{2} - 2\right) - 8 = 0$$
$$3p - p - 8 + 16 = 0$$
$$\therefore p = -4 \textbf{ (shown)}$$

□

(b)

$$\boldsymbol{C(2, -1)}$$

(c) From the tangent $x = -8$,

$$\text{Radius of circle} = 10 \text{ units}$$

$$\therefore 10 = \sqrt{(2)^2 + (-1)^2 - k}$$
$$100 = 4 + 1 - k$$
$$k = \mathbf{-95}$$

(d)

$$\text{Length of } CA = \sqrt{(2-14)^2 + (-1-(-8))^2}$$
$$= \sqrt{193} > 10$$

Since the distance between $(14, -8)$ and the centre is bigger than the radius, therefore the point $(14, -8)$ lies **outside** the circle

(e)

***ACX* is a straight line**

3. (a) Since the centres lie on the line $y = x$, let the centres of C_1 and C_2 be (a, a)

$$\begin{aligned} a^2 + (a+3)^2 &= 5 \\ a^2 + a^2 + 6a + 9 - 5 &= 0 \\ a^2 + 3a + 2 &= 0 \\ (a+1)(a+2) &= 0 \end{aligned}$$

$$a = -1 \qquad \text{or} \qquad a = -2$$

$$C_1 : \mathbf{(x+1)^2 + (y+1)^2 = 5} \qquad \text{and} \qquad C_2 : \mathbf{(x+2)^2 + (y+2)^2 = 5}$$

(b) For C_1, substitute $y = 0$,

$$(x+1)^2 + (1)^1 = 5 \text{ ......(1)}$$

For C_2, substitute $y = 0$,

$$(x+2)^2 + (2)^2 = 5 \text{ ......(2)}$$

Let Equation (1) = Equation (2),

$$\begin{aligned} (x+1)^2 + 1 &= (x+2)^2 + 4 \\ x^2 + 2x + 1 + 1 &= x^2 + 4x + 4 + 4 \\ 2x + 6 &= 0 \\ x &= \mathbf{-3} \end{aligned}$$

(c)

$$\begin{aligned} \text{Distance between 2 centres} &= \sqrt{(-1+2)^2 + (-1+2)^2} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{Greatest distance} &= \sqrt{5} + \sqrt{2} + \sqrt{5} \\ &= \sqrt{2} + 2\sqrt{5} \\ &= 5.886349... \\ &= \mathbf{5.89 \text{ units}} \end{aligned}$$

4. (a)

$$\begin{aligned}\text{Gradient of } PQ &= \frac{7-3}{6-(-2)}\\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\text{Gradient of } RQ &= \frac{11-1}{4-6}\\ &= -2\end{aligned}$$

$$\begin{aligned}\text{Gradient of } PQ \times \text{Gradient of } RQ &= \frac{1}{2}\times(-2)\\ &= -1\end{aligned}$$

Since the product of the gradients is -1, PQ is perpendicular to RQ. Hence, $\angle PQR = 90^\circ$

□

(b) Using the property: **angles in a semicircle**, PR is the hypotenuse of the right-triangle PQR, and hence, P, Q and R lie on the circle with diameter PR

(c)

$$\begin{aligned}\text{Centre} &= \text{Midpoint of } PR\\ &= \left(\frac{-2+4}{2}, \frac{3+11}{2}\right)\\ &= (1,7)\end{aligned}$$

$$\begin{aligned}\text{Radius} &= PC\\ &= \sqrt{(1-(-2))^2+(7-3)^2}\\ &= 5 \text{ units}\end{aligned}$$

$$\therefore \boldsymbol{(x-1)^2+(y-7)^2=25}$$

(d)

$$\begin{aligned}\text{Distance} &= \sqrt{(3-1)^2+(2-7)^2}\\ &= \sqrt{29} > 5\end{aligned}$$

Since the distance between $(3,2)$ and the centre is bigger than the radius, therefore the point $(3,-2)$ lies **outside** the circle

(e) Note that the centre lies on the normal to the circle. Hence, substitute $(1,7)$ into the equation of the circle

$$\begin{aligned}3(7)-4(1) &= k\\ k &= \mathbf{17}\end{aligned}$$

10 Linear Law

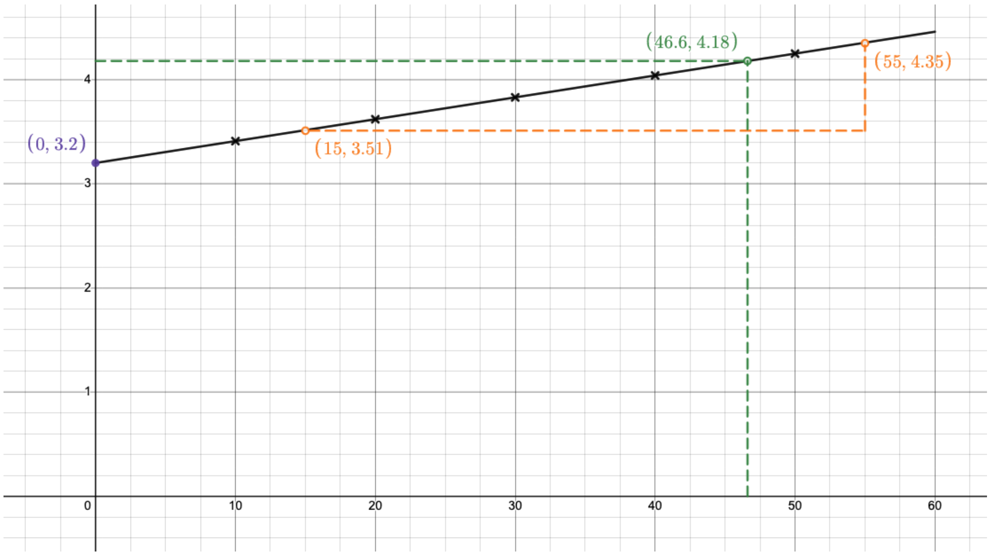
10.1 Full Solutions

1. (a)

$$y = k(2)^{\frac{t}{m}}$$
$$\lg y = \lg k + \frac{t}{m}\lg 2$$
$$\lg y = \left(\frac{1}{m}\lg 2\right)t + \lg k$$

Hence, we are plotting $\lg y$ against t

t	10	20	30	40	50
$\lg y$	3.41	3.63	3.85	4.06	4.28



(b) (i) From the graph,

$$\lg k = 3.2$$
$$k = 10^{3.2}$$
$$= 1584.893192...$$
$$= \mathbf{1580\ (3.s.f.)}$$

$$\frac{1}{m}\lg 2 = \frac{4.35 - 3.51}{55 - 15}$$
$$= 0.021$$
$$m = \frac{1}{\left(\frac{0.021}{\lg 2}\right)}$$
$$= 14.334761...$$
$$= \mathbf{14.3\ (3.s.f.)}$$

(ii) When $y = 15000$,

$$\lg y = 4.18\ (3.s.f.)$$

From the graph,

$$t = \mathbf{46.6\ minutes}$$

2. (a)

$$\begin{aligned}\text{Gradient} &= \frac{9-3}{2-5} \\ &= -2\end{aligned}$$

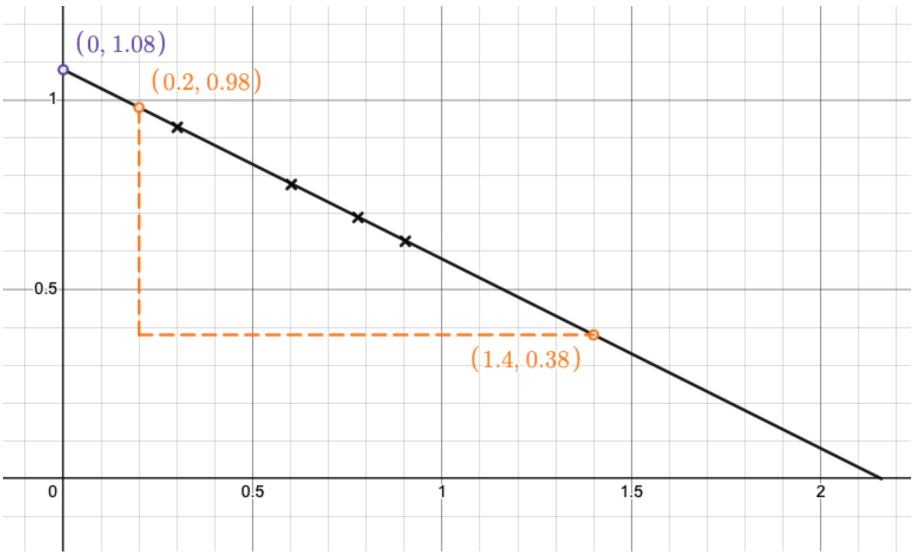
$$\begin{aligned}\therefore \frac{x}{y} - 3 &= -2\left(\frac{1}{x} - 5\right) \\ \frac{x}{y} &= -\frac{2}{x} + 13 \\ \frac{x}{y} &= \frac{13x-2}{x}\end{aligned}$$

$$\therefore \boldsymbol{y = \frac{x^2}{13x-2}}$$

(b)

$$\begin{aligned}&x^n y = k \\ &\lg y = (-n)\lg x + \lg k\end{aligned}$$

$\lg x$	0.301	0.602	0.778	0.903
$\lg y$	0.928	0.777	0.690	0.627



From the graph,

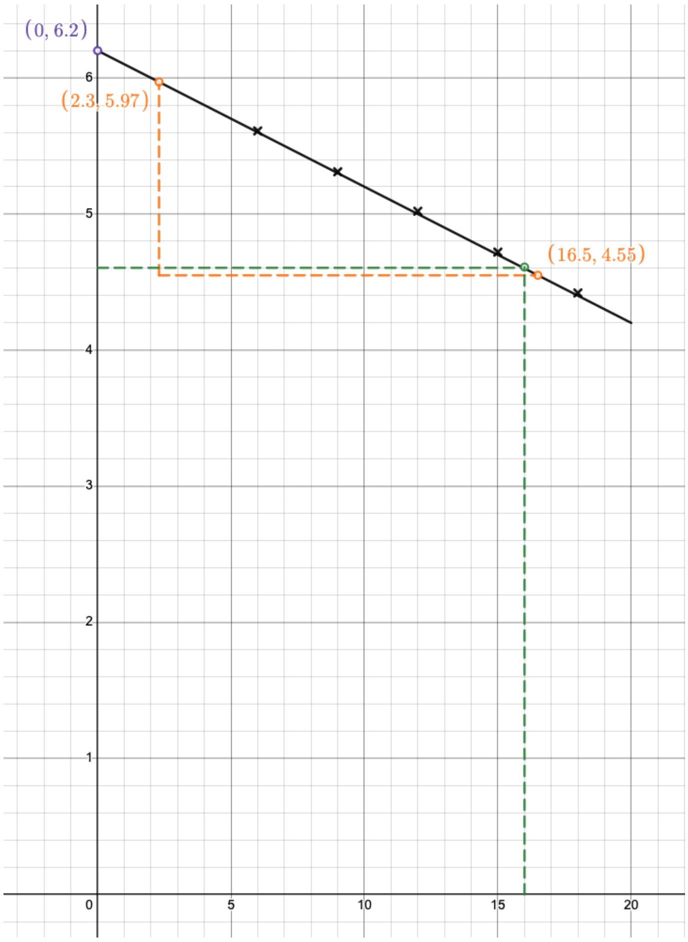
$$\begin{aligned}\lg k &= 1.08 \\ k &= 10^{1.08} \\ &= 12.022644.. \\ &= \mathbf{12.0\ (3.s.f.)}\end{aligned}$$

$$\begin{aligned}-n &= \frac{0.98-0.38}{0.2-1.4} \\ n &= \mathbf{\frac{1}{2}}\end{aligned}$$

3. (a)

$$P = P_0 e^{-kt}$$
$$\ln P = \ln P_0 e^{-kt}$$
$$\ln P = -kt + \ln P_0$$

t	6	9	12	15	18
$\ln P$	5.61	5.31	5.02	4.72	4.42



(b) From the graph

$$\ln P_0 = 6.2$$
$$P_0 = e^{6.2}$$
$$= 492.749041...$$
$$\approx \mathbf{500\ (nearest\ hundredth)}$$

$$-k = \frac{5.97 - 4.55}{2.3 - 16.5}$$
$$k = \mathbf{\frac{1}{10}}$$

(c) When $P = 100$,

$$\ln P = 4.61\ (3.s.f.)$$

From the graph,

$$\text{Number of years} = 16.5 \approx \mathbf{17\ years}$$

11 Proofs of Plane Geometry

11.1 Full Solutions

1. (a) Since D and G are the mid-points of HB and AB respectively

$$\therefore GO \text{ is parallel to } AD \text{ (midpoint theorem)}$$

$$\angle DAB = 90^\circ \text{ (angles in a semi-circle)}$$

$$\therefore \angle GOB = 90^\circ \text{ (corresponding angles)}$$

□

(b)

$$\angle DAH = \angle CAD \ (AD \text{ bisects } \angle CAH)$$

$$\angle ABD = \angle DAH \text{ (alternate segment theorem)}$$

$$\angle CBD = \angle CAD \text{ (angles in the same segment)}$$

$$\therefore \angle CBD = \angle DAH = \angle ABD \textbf{ (shown)}$$

□

2. (a)

$$\begin{aligned} \angle DBF &= \angle BAD \text{ (alternate segment theorem)} \\ &= \angle ADB \ (\triangle ABD \text{ is an isosceles triangle}) \end{aligned}$$

$$AD \text{ is parallel to } BF \text{ (alternate angles)}$$

$$\text{Since } AD = BF,\ ABFD \text{ is a parallelogram}$$

□

(b)

$$\angle EDF = \angle DBC \text{ (alternate segment theorem) (A)}$$

$$\begin{aligned} \angle DFE &= 180^\circ - \angle BFD \text{ (adjacent angles on a straight line)} \\ &= 180^\circ - \angle BAD \text{ (opposite angles in a parallelogram)} \\ &= 180^\circ - (180^\circ - \angle DCB) \text{ (angles in opposite segment)} \\ &= \angle DCB \text{ (A)} \end{aligned}$$

$$\text{By the AA similarity test, } \triangle BCD \text{ is similar to } \triangle DFE$$

□

(c) From part (b),

$$\frac{BD}{DE} = \frac{CD}{EF}$$

$$BD \times EF = CD \times DE \textbf{ (shown)}$$

□

3. (a)

$$\angle BDC = 90^\circ \text{ (angles in a semicircle)}$$

$$\angle BFC = 90^\circ \text{ (angles in the same segment)}$$

$$\begin{aligned}\angle BFA &= 180^\circ - \angle BFE \text{ ($AFEC$ is a straight line)}\\ &= 180^\circ - 90^\circ\\ &= 90^\circ \text{ (angles on a straight line)}\end{aligned}$$

$$\angle BHA = \angle BFA = 90^\circ \text{ (angles in the same segment)}$$

$$\begin{aligned}\angle AHD &= 180^\circ - \angle BHA \text{ ($BHED$ is a straight line)}\\ &= 180^\circ - 90^\circ\\ &= 90^\circ \text{ (angles on a straight line)}\end{aligned}$$

$$\angle AHD = \angle BDC = \angle HDC \text{ (alternate angles)}$$

$$\therefore CD \text{ is parallel to } AH \textbf{ (shown)}$$

□

(b)

$$\angle BHA = \angle BFA = 90^\circ \text{ (angles in the same segment)}$$

$$\therefore \text{Using angles in a semicircle, } AB \text{ is the diameter of the circle } \textbf{(shown)}$$

□

(c) Since AB and BC are tangential to the smaller and bigger circle respectively

$$\angle ABC = 90^\circ \text{ (tangent is perpendicular to radius)}$$

$$\angle BFC = 90^\circ \text{ (part (a))}$$

$$\therefore \angle ABC = \angle BFC \text{ (A)}$$

$$\angle BCA = \angle FCB \text{ (common angle) (A)}$$

$$\text{By the AA similar test, } \triangle ABC \text{ is similar to } \triangle BFC$$

□

(d) From part (c),

$$\begin{aligned}\frac{BC}{FC} &= \frac{AC}{CB}\\ BC^2 &= CF \times AC \text{ ......(1)}\end{aligned}$$

Since $\triangle ABC$ is a right-triangle, by Pythagoras Theorem,

$$BC^2 = AC^2 - AB^2 \text{ ......(2)}$$

Hence, let Equation (1) = Equation (2),

$$\therefore AC^2 - AB^2 = CF \times AC \textbf{ (shown)}$$

□

4. (a)

$$\angle ZXQ = \angle SRX \text{ (alternate segment theorem)}$$

$$\angle ZXQ = \angle QXR \text{ (} XQ \text{ is the angle bisector of } \angle RXZ\text{)}$$

$$\therefore \angle QXR = \angle SRX$$

$$\therefore SR = SX \text{ (base angles of an isosceles triangle)}$$

(b) Let $\angle QXR = x$

$$\angle RSX = 180^\circ - 2x \text{ (angles in an isosceles triangle)}$$

$$\angle YSQ = 180^\circ - 2x \text{ (vertically opposite angles)}$$

$$\angle RZX = \angle ZXR = 2x \text{ (base angles of the isosceles triangle)}$$

$$\begin{aligned} \therefore \angle RZX + \angle YSQ &= 180^\circ - 2x + 2x \\ &= 180^\circ \end{aligned}$$

Using opposite angles are supplementary in a cyclic quadrilateral, Z, Y, S and Q can have a circle drawn through **(shown)**

□

12 Differentiation

12.1 Full Solutions

1. (a)

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sqrt{1-4x}\left(2e^{2x}\right) - \left[\frac{1}{2}(1-4x)^{-\frac{1}{2}}(-4)\right]\left(e^{2x}\right)}{\left(\sqrt{1-4x}\right)^2} \\ &= \frac{e^{2x}\left(2\sqrt{1-4x} + 2(1-4x)^{-\frac{1}{2}}\right)}{1-4x} \\ &= \frac{2e^{2x}(1-4x+1)}{(1-4x)\sqrt{1-4x}} \\ &= \frac{4e^{2x}(1-2x)}{(1-4x)\sqrt{1-4x}} \ \textbf{(shown)}\end{aligned}$$

□

(b) (i)

$$\begin{aligned}\frac{dy}{dx} &= 1 + 2\sin x\cos x \\ &= 1 + \sin 2x\end{aligned}$$

(ii) At stationary point, $\frac{dy}{dx} = 0$

$$\begin{aligned}1 + \sin 2x &= 0 \\ \sin 2x &= -1\end{aligned}$$

$$\begin{aligned}\alpha &= \sin^{-1}(1) \\ &= \frac{\pi}{2} \quad \text{(Quadrant 3 or 4)}\end{aligned}$$

$$\begin{aligned}x &= \frac{\left(\pi + \frac{\pi}{2}\right)}{2} \\ &= \frac{3\pi}{4}\end{aligned}$$

Substitute $x = \frac{3\pi}{4}$ into the curve,

$$\begin{aligned}y &= \frac{3\pi}{4} + \sin^2\left(\frac{3\pi}{4}\right) \\ &= \frac{3\pi+2}{4} \\ \therefore &\left(\boldsymbol{\frac{3\pi}{4}, \frac{3\pi+2}{4}}\right)\end{aligned}$$

(c)

$$
\begin{aligned}
y &= \ln\left(\frac{x-2}{x-3}\right)^2 \\
&= 2\left[\ln(x-2) - \ln(x-3)\right]
\end{aligned}
$$

$$
\begin{aligned}
\therefore \frac{dy}{dx} &= 2\left(\frac{1}{x-2} - \frac{1}{x-3}\right) \\
&= -\frac{2}{(x-2)(x-3)}
\end{aligned}
$$

Since the graph is decreasing, $\frac{dy}{dx} < 0$

$$
\begin{aligned}
-\frac{2}{(x-2)(x-3)} &< 0 \\
(x-2)(x-3) &> 0
\end{aligned}
$$

$$
\therefore \boldsymbol{x < 2} \quad \text{and} \quad \boldsymbol{x > 3}
$$

2. (a)

$$\begin{aligned}\frac{dy}{dx} &= \left(3 - 10x + \frac{1}{x}\right) e^{3x-5x^2+\ln 2} \\ &= \left(3 - 10x + \frac{1}{x}\right) 2xe^{3x-5x^2} \\ &= \mathbf{2e^{3x-5x^2}\left(-10x^2 + 3x + 1\right)}\end{aligned}$$

(b) At the stationary point, $\frac{dy}{dx} = 0$,

$$2e^{3x-5x^2}\left(-10x^2 + 3x + 1\right) = 0$$

$$2e^{3x-5x^2} = 0 \text{ (N.A.)} \qquad \text{or} \qquad -10x^2 + 3x + 1 = 0$$

For the quadratic expression,

$$\begin{aligned}-10x^2 + 3x + 1 &= 0 \\ (2x-1)(-5x-1) &= 0\end{aligned}$$

$$\therefore x = \frac{1}{2} \qquad \text{or} \qquad x = -\frac{1}{5} \text{ (rej)}$$

Hence, substitute $x = \frac{1}{2}$ into the curve,

$$\begin{aligned}\therefore y &= e^{3\left(\frac{1}{2}\right)-5\left(\frac{1}{2}\right)^2+\ln 2\left(\frac{1}{2}\right)} \\ &= e^{\frac{1}{4}}\end{aligned}$$

$$\therefore \boldsymbol{\left(\frac{1}{2}, e^{\frac{1}{4}}\right)}$$

(c)

$$\begin{aligned}\frac{d^2y}{dx^2} &= \left[2\left(3-10x\right)e^{3x-5x^2}\right]\left(-10x^2+3x+1\right) + (-20x+3)\left(2e^{3x-5x^2}\right) \\ &= 2e^{3x-5x^2}\left[(3-10x)\left(-10x^2+3x+1\right) + (-20x+3)\right]\end{aligned}$$

$$\begin{aligned}\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} &= 2e^{3\left(\frac{1}{2}\right)-5\left(\frac{1}{2}\right)^2}\left[\left(3-10\left(\frac{1}{2}\right)\right)\left(-10\left(\frac{1}{2}\right)^2+3\left(\frac{1}{2}\right)+1\right)-20\left(\frac{1}{2}\right)+3\right] \\ &= -17.976355... < 0\end{aligned}$$

$$\therefore \left(\frac{1}{2}, e^{\frac{1}{4}}\right) \text{ is a } \textbf{maximum} \text{ point}$$

3. (a)

$$y = he^x + \frac{k}{e^{2x}}$$

$$\frac{dy}{dx} = he^x - \frac{2k}{e^{2x}} \qquad \frac{d^2y}{dx^2} = he^x + \frac{4k}{e^{2x}}$$

$$\begin{aligned} \text{LHS} &= \frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right) \\ &= he^x + \frac{4k}{e^{2x}} - 2\left(he^x - \frac{2k}{e^{2x}}\right) \\ &= -he^x + \frac{8k}{e^{2x}} \end{aligned}$$

$$\therefore h = \mathbf{-1} \qquad k = \mathbf{\frac{1}{4}}$$

(b) Let the total surface area of ice block be A

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r(2r) \\ &= 6\pi r^2 \end{aligned}$$

$$\begin{aligned} \frac{dr}{dt} &= \frac{dr}{dA} \times \frac{dA}{dt} \\ &= \frac{1}{\left(\frac{dA}{dr}\right)} \times \frac{dA}{dt} \\ &= \frac{1}{12\pi r} \times (-72) \\ &= -\frac{6}{\pi r} \end{aligned}$$

Hence, when $r = 5$,

$$\left.\frac{dr}{dt}\right|_{r=5} = -\frac{6}{5\pi}$$

$\therefore$ The radius of the ice block decreases at $\boldsymbol{\frac{6}{5\pi}}$ **cm/s**

4. (a)

$$2x^2 + 2(2x + x)h = 2700$$

$$\therefore \boldsymbol{h = \frac{1350 - x^2}{3x}}$$

(b)

$$\begin{aligned} V &= 2x^2 h \\ &= 2x^2 \left(\frac{1350 - x^2}{3x} \right) \\ &= 900x - \frac{2}{3}x^3 \textbf{ (shown)} \end{aligned}$$

□

(c)

$$\frac{dV}{dx} = 900 - 2x^2$$

When V is maximum, $\frac{dV}{dx} = 0$,

$$\begin{aligned} 900 - 2x^2 &= 0 \\ x^2 &= 450 \\ x &= \boldsymbol{15\sqrt{2}} \quad \text{(rej -ve)} \end{aligned}$$

$$\begin{aligned} \left.\frac{d^2y}{dx^2}\right|_{x=15\sqrt{2}} &= -4x \\ &= -4\left(15\sqrt{2}\right) \\ &= -60\sqrt{2} < 0 \end{aligned}$$

Hence, V is maximum

(d)

$$\begin{aligned} V &= 900\left(15\sqrt{2}\right) - \frac{2}{3}\left(15\sqrt{2}\right)^3 \\ &= \boldsymbol{9000\sqrt{2}} \textbf{ cm}^{\boldsymbol{3}} \end{aligned}$$

5. (a) (i)

$$\begin{aligned}\frac{dy}{dx} &= (3x)\left(-2e^{-2x}\right) + \left(e^{-2x}\right)(3)\\ &= \mathbf{3e^{-2x}(-2x+1)}\end{aligned}$$

(ii)

$$\begin{aligned}\frac{d^2y}{dx^2} &= 3e^{-2x}(-2) + (-2x+1)\left(3e^{-2x}\right)(-2)\\ &= -6e^{-2x}(2-2x)\\ &= 12e^{-2x}(x-1)\end{aligned}$$

$$\begin{aligned}\therefore p &= e^{2x}\left(\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y\right)\\ &= e^{2x}\left[\left(12xe^{-2x} - 12e^{-2x}\right) + \left(3e^{-2x} - 6xe^{-2x}\right) - 2\left(3xe^{-2x}\right)\right]\\ &= 12x - 12 + 3 - 6x - 6x\\ &= \mathbf{-9}\end{aligned}$$

(b) (i)

$$\begin{aligned}y &= \ln\left(\frac{1-\cos x}{\sin x}\right)\\ &= \ln(1-\cos x) - \ln(\sin x)\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin x}{1-\cos x} - \frac{\cos x}{\sin x}\\ &= \frac{\sin^2 x - \cos x(1-\cos x)}{\sin x(1-\cos x)}\\ &= \frac{\sin^2 x - \cos x + \cos^2 x}{\sin x(1-\cos x)}\\ &= \frac{1-\cos x}{\sin x(1-\cos x)}\\ &= \frac{1}{\sin x}\\ &= \csc x \ \textbf{(shown)}\end{aligned}$$

□

(ii)

$$\begin{aligned}\frac{dy}{dt} &= 2\left(\frac{dx}{dt}\right)\\ \frac{dy}{dt} &= \left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right)\\ \therefore 2\left(\frac{dx}{dt}\right) &= \left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right)\\ \frac{dy}{dx} &= 2\end{aligned}$$

$$\begin{aligned}\csc x &= 2\\ \sin x &= \frac{1}{2}\\ x &= \boldsymbol{\frac{\pi}{6}} \ \textbf{rad}\end{aligned}$$

13 Integration

13.1 Full Solutions

1.

$$\begin{aligned} f'(x) &= \int 4e^{2x} + \frac{9}{(3x+1)^2}\ dx \\ &= 2e^{2x} + \frac{9(3x+1)^{-1}}{(-1)(3)} + c \\ &= 2e^{2x} - \frac{3}{3x-1} + c \end{aligned}$$

Since $f'(0) = -1$,

$$\begin{aligned} f'(0) &= -1 \\ 2 - 3 + c &= -1 \\ c &= 0 \end{aligned}$$

$$f'(x) = 2e^{2x} - \frac{3}{3x+1}$$

$$\begin{aligned} f(x) &= \int 2e^{2x} - \frac{3}{3x+1}\ dx \\ &= e^{2x} - \ln(3x+1) + d \end{aligned}$$

Since $f(0) = 2$,

$$\begin{aligned} 1 - \ln(1) + d &= 2 \\ d &= 1 \end{aligned}$$

$$f(x) = \boldsymbol{e^{2x} - \ln(3x+1) + 1}$$

2. (a) At A, $x = 3$

$$\left.\frac{dy}{dx}\right|_{x=3} = \frac{1}{3}e^{\frac{1}{3}x}$$
$$= \frac{1}{3}e^{\frac{1}{3}(3)}$$
$$= \frac{1}{3}e$$

When $x = 3$,

$$y = e^{\frac{1}{3}(3)} + 2$$
$$= 2 + e$$

$$\therefore A(3, 2 + e)$$

Hence, the equation of the tangent is:

$$y - (2 + e) = \frac{1}{3}e(x - 3)$$
$$y = \frac{1}{3}ex + 2$$

$$\therefore B(0, 2)$$

Hence,

$$\text{Area under the graph} = \int_0^3 e^{\frac{1}{3}x} + 2 \; dx - \frac{1}{2}(2 + 2 + e)(3)$$
$$= \left[3e^{\frac{1}{3}x} + 2x\right]_0^3 - \frac{3}{2}(e + 4)$$
$$= \left[3e^{\frac{1}{3}(3)} + 2(3)\right] - \left[3e^{\frac{1}{3}(0)} + 2(0)\right] - \frac{3}{2}e - 6$$
$$= 3e + 6 - 3 - \frac{3}{2}e - 6$$
$$= \boldsymbol{\left(\frac{3}{2}e - 3\right)} \textbf{ units}^{\mathbf{2}}$$

(b) When $x = 0$,

$$y = e^{\frac{1}{3}(0)} + 2$$
$$= 3$$

Gradient of the tangent when $x = 0$,

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{1}{3}e^{\frac{1}{3}(0)}$$
$$= \frac{1}{3}$$

$$\text{Gradient of normal} = -3$$

$$\therefore y - 3 = -3(x - 0)$$

$$\boldsymbol{y = -3x + 3}$$

3.

$$y = A - B\cos 4x - \frac{1}{2}\sin 2x$$

$$\frac{dy}{dx} = 4B\sin 4x - \cos 2x \qquad \frac{d^2y}{dx^2} = 16B\cos 4x + 2\sin 2x$$

$$\begin{aligned}\therefore \frac{d^2y}{dx^2} + 4y &= 16B\cos 4x + 2\sin 2x + 4\left[A - B\cos 4x - \frac{1}{2}\sin 2x\right] \\ &= 12B\cos 4x + 4A\end{aligned}$$

Hence, comparing coefficients,

$$\boldsymbol{A = \frac{1}{4} \qquad B = \frac{1}{4}}$$

4. (a)

$$\begin{aligned}\int_0^5 f(x)\ dx &= \int_0^2 f(x)\ dx + \int_2^5 f(x)\ dx \\ &= 4 + 12 \\ &= \mathbf{16}\end{aligned}$$

(b)

$$\begin{aligned}\int_0^2 \left[f(x) + mx^2\right]\ dx &= \int_5^2 f(x)\ dx \\ \int_0^2 f(x)\ dx + \int_0^2 mx^2\ dx &= -\int_2^5 f(x)\ dx \\ 4 + \left[\frac{1}{3}mx^3\right]_0^2 &= -12 \\ 4 + \left[\frac{8}{3}m - 0\right] &= -12 \\ m &= \mathbf{-6}\end{aligned}$$

5. (a) At P, $y = 0$,

$$\begin{aligned} 0 &= \frac{2x+4}{x-1} \\ x &= -2 \end{aligned}$$

$$\boldsymbol{P(-2, 0)}$$

At Q, $x = 0$,

$$\begin{aligned} y &= \frac{2(0)+4}{(0)-1} \\ &= -4 \end{aligned}$$

$$\boldsymbol{Q(0, -4)}$$

(b)

$$y = \frac{2x+4}{x-1} \text{ ......(1)}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x-1)(2)-(2x+4)(1)}{(x-1)^2} \\ &= \frac{2x-2-2x-4}{(x-1)^2} \\ &= -\frac{6}{(x-1)^2} \end{aligned}$$

At P, $x = -2$,

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{x=-2} &= -\frac{6}{(-2-1)^2} \\ &= -\frac{2}{3} \end{aligned}$$

$$\therefore \text{Gradient of normal} = \frac{3}{2}$$

Hence, the equation of the normal is

$$\begin{aligned} y - 0 &= \frac{3}{2}(x+2) \\ y &= \frac{3}{2}x + 3 \text{ ......(2)} \end{aligned}$$

$$\therefore \boldsymbol{R(0, 3)}$$

To find S, let Equation (1) = Equation (2),

$$\begin{aligned}\frac{2x+4}{x-1} &= \frac{3}{2}x+3\\ 4x+8 &= (3x+6)(x-1)\\ 3x^2-3x+6x-6-4x-8 &= 0\\ 3x^2-x-14 &= 0\\ (3x-7)(x+2) &= 0\end{aligned}$$

$$\therefore x = \frac{7}{3} \qquad \text{or} \qquad x = -2 \text{ (N.A.)}$$

Substitute $x = \frac{7}{3}$ into Equation (2),

$$\begin{aligned} y &= \frac{3}{2}\left(\frac{7}{3}\right)+3\\ &= 6\frac{1}{2}\end{aligned}$$

$$\therefore \boldsymbol{S\left(2\frac{1}{3}, 6\frac{1}{2}\right)}$$

(c) We first breakdown the equation of the curve using long division (or any appropriate methods)

$$\frac{2x+4}{x-1} = 2+\frac{6}{x-1}$$

$$\begin{aligned}\therefore \text{Shaded region} &= \frac{1}{2}\left(3+6\frac{1}{2}\right)\left(2\frac{1}{3}\right)+\int_{2\frac{1}{3}}^{3}\frac{2x+4}{x-1}\,dx\\ &= 11\frac{1}{12}+\int_{2\frac{1}{3}}^{3} 2+\frac{6}{x-1}\,dx\\ &= 11\frac{1}{12}+\left[2x+6\ln(x-1)\right]_{2\frac{1}{3}}^{3}\\ &= 14.849457...\\ &= \mathbf{14.8\ units^2\ (3.s.f.)}\end{aligned}$$

14 Differentiation & Integration

14.1 Full Solutions

1. (a)

$$\begin{aligned}\frac{d}{dx}\left(\tan^3 x\right) &= 3\tan^2 x\left(\sec^2 x\right)\\ &= 3\left(\sec^2 x - 1\right)\left(\sec^2 x\right)\\ &= 3\sec^4 x - 3\sec^2 x \ \textbf{(shown)}\end{aligned}$$

□

(b)

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sec^4 x - 2\sec^2 x\ dx &= \frac{1}{3}\int_0^{\frac{\pi}{4}} 3\sec^4 x - 3\sec^2 x\ dx - \int_0^{\frac{\pi}{4}} \sec^2 x\ dx\\ &= \frac{1}{3}\left[\tan^3 x\right]_0^{\frac{\pi}{4}} - \left[\tan x\right]_0^{\frac{\pi}{4}}\\ &= \frac{1}{3}\left[1-0\right] - \left[1-0\right]\\ &= -\boldsymbol{\frac{2}{3}}\end{aligned}$$

2. (a)

$$\frac{2x^3-20x^2-17x-10}{(x^2-4)(2x^2+1)} = \frac{2x^3-20x^2-17x-10}{(x-2)(x+2)(2x^2+1)} = \frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx+D}{2x^2+1}$$

$$\therefore 2x^3-20x^2-17x-10 = A(x+2)\left(2x^2+1\right)+B(x-2)\left(2x^2+1\right)+(Cx+D)(x-2)(x+2)$$

Let $x=2$,

$$2(2)^3-20(2)^2-17(2)-10 = A(2+2)\left(2(2)^2+1\right)$$
$$A=-3$$

Let $x=-2$,

$$2(-2)^3-20(-2)^2-17(-2)-10 = B(-2-2)\left(2(-2)^2+1\right)$$
$$B=2$$

Let $x=0$,

$$2(0)^3-20(0)^2-17(0)-10 = -3(0+2)\left(2(0)^2+1\right)+2(0-2)\left(2(0)^2+1\right)+D(0-2)(0+2)$$
$$D=0$$

Let $x=1$,

$$2(1)^3-20(1)^2-17(1)-10 = -3(1+2)\left(2(1)^2+1\right)+2(1-2)\left(2(1)^2+1\right)+C(1-2)(1+2)$$
$$C=4$$

$$\therefore \mathbf{\frac{2x^3-20x^2-17x-10}{(x^2-4)(2x^2+1)} = -\frac{3}{x-2}+\frac{2}{x+2}+\frac{4x}{2x^2+1}}$$

(b)

$$\frac{d}{dx}\left[\ln\left(2x^2+1\right)\right] = \mathbf{\frac{4x}{2x^2+1}}$$

(c)

$$\int \frac{2x^3-20x^2-17x-10}{(x^2-4)(2x^2+1)}\,dx = \int -\frac{3}{x-2}+\frac{2}{x+2}+\frac{4x}{2x^2+1}\,dx$$
$$= \mathbf{-3\ln(x-2)+2\ln(x+2)+\ln\left(2x^2+1\right)+c}$$

3. (a)

$$y=(x+3)\sqrt{2x-3}$$

$$\frac{dy}{dx} = \sqrt{2x-3}+\frac{1}{2}(2x-3)^{-\frac{1}{2}}(2)(x+3)$$
$$= \sqrt{2x-3}+\frac{x+3}{\sqrt{2x-3}}$$
$$= \frac{2x-3+x+3}{\sqrt{2x-3}}$$
$$= \mathbf{\frac{3x}{\sqrt{2x-3}}}$$

(b)

$$\int \frac{x}{\sqrt{2x-3}}\,dx = \frac{1}{3}\int \frac{3x}{\sqrt{2x-3}}\,dx$$
$$= \mathbf{\frac{1}{3}(x+3)\sqrt{2x-3}+c}$$

4.

$$f''(x) = 24\sin 4x - 12\cos 4x$$

$$\begin{aligned} f'(x) &= \int 24\sin 4x - 12\cos 2x \ dx \\ &= \frac{-24\cos 4x}{4} - \frac{12\sin 2x}{2} + c \\ &= -6\cos 4x - 6\sin 2x + c \end{aligned}$$

Let $f'\left(\frac{\pi}{4}\right) = 0,$

$$\begin{aligned} -6\cos\left[4\left(\frac{\pi}{4}\right)\right] - 6\sin\left[2\left(\frac{\pi}{4}\right)\right] + c &= 0 \\ 6 - 6 + c &= 0 \\ c &= 0 \end{aligned}$$

$$\therefore f'(x) = -6\cos 4x - 6\sin 2x$$

$$\begin{aligned} f(x) &= \int -6\cos 4x - 6\sin 2x \ dx \\ &= \frac{-6\sin 4x}{4} + \frac{6\cos 2x}{2} + d \\ &= -\frac{3}{2}\sin 4x + 3\cos 2x + d \end{aligned}$$

Let $f\left(\frac{\pi}{4}\right) = 1$

$$\begin{aligned} -\frac{3}{2}\sin\left[4\left(\frac{\pi}{4}\right)\right] + 3\cos\left[2\left(\frac{\pi}{4}\right)\right] + d &= 1 \\ c &= 1 \end{aligned}$$

$$\therefore f(x) = -\frac{3}{2}\sin 4x + 3\cos 2x + 1$$

Hence,

$$\begin{aligned} f''(x) + 4f(x) &= 24\sin 4x - 12\cos 2x + 4\left[-\frac{3}{2}\sin 4x + 3\cos 2x + 1\right] \\ &= 24\sin 4x - 12\cos 2x - 6\sin 4x + 12\cos 2x + 4 \\ &= 18\sin 4x + 4 \end{aligned}$$

$$\therefore k = 18 \qquad p = 4 \qquad q = 4$$

15 Kinematics

15.1 Full Solutions

1. (a)

$$v = \frac{27}{2(3t+1)^2} - \frac{3t+1}{2}$$

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= -\frac{81}{(3t+1)^3} - \frac{3}{2} \end{aligned}$$

Initially, $t = 0$,

$$\begin{aligned} a &= -\frac{81}{(3(0)+1)^3} - \frac{3}{2} \\ &= \mathbf{-82\frac{1}{2}\ m/s^2} \end{aligned}$$

(b) For all $t > 0$

$$\frac{dv}{dt} = -\frac{81}{(3t+1)^3} - \frac{3}{2} < 0$$

$\therefore$ Velocity is **decreasing**

(c) We shall first test for any instantaneous rest, $v = 0$

$$\begin{aligned} \frac{27}{2(3t+1)^2} &= \frac{3t+1}{2} \\ (3t+1)^3 &= 27 \\ 3t+1 &= 3 \\ t &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} s &= \int \frac{27}{2(3t+1)^2} - \frac{3t+1}{2}\ dt \\ &= \int \frac{27}{2}(3t+1)^{-2} - \frac{3}{2}t - \frac{1}{2}\ dt \\ &= \frac{27}{2}\left[\frac{(3t+1)^{-1}}{(3)(-1)}\right] - \frac{3}{4}t^2 - \frac{1}{2}t + c \\ &= -\frac{9}{2(3t+1)} - \frac{3}{4}t^2 - \frac{1}{2}t + c \end{aligned}$$

When $t = 0$, $s = 0$,

$$c = \frac{9}{2}$$

$$\therefore s = -\frac{9}{2(3t+1)} - \frac{3}{4}t^2 - \frac{1}{2}t + \frac{9}{2}$$

When $t = \frac{2}{3}$,

$$\begin{aligned} s &= -\frac{9}{2\left[3\left(\frac{2}{3}\right)+1\right]} - \frac{3}{4}\left(\frac{2}{3}\right)^2 - \frac{1}{2}\left(\frac{2}{3}\right) + \frac{9}{2} \\ &= 2\frac{1}{3} \text{ m} \end{aligned}$$

When $t = 6$,

$$\begin{aligned} s &= -\frac{9}{2\left[3(6)+1\right]} - \frac{3}{4}(6)^2 - \frac{1}{2}(6) + \frac{9}{2} \\ &= -25\frac{14}{19} \text{ m} \end{aligned}$$

Hence,

$$\begin{aligned} \text{Total distance travelled} &= 2\frac{1}{3} + 2\frac{1}{3} + 25\frac{14}{19} \\ &= 30\frac{23}{57} \end{aligned}$$

$$\begin{aligned} \therefore \text{Average speed} &= \frac{\left(30\frac{23}{57}\right)}{6} \\ &= 5.067251... \\ &= \mathbf{5.07 \text{ m/s } (3.s.f.)} \end{aligned}$$

2. (a) When $t = 0$,

$$v = 10e^{-2(0)} - 3$$
$$= \mathbf{7\ m/s}$$

(b)

$$v = 10e^{-2t} - 3$$

$$a = \frac{dv}{dt}$$
$$= -20e^{-2t}$$

When $t = 1$,

$$a = -20e^{-2(1)}$$
$$= -2.706705...$$
$$= \mathbf{-2.71\ m/s^2\ (3.s.f.)}$$

(c) At instantaneous rest, $v = 0$,

$$10e^{-2t} - 3 = 0$$
$$e^{-2t} = \frac{3}{10}$$

$$t = -\frac{1}{2}\ln\left(\frac{3}{10}\right)$$
$$= 0.601986...$$
$$= \mathbf{0.602\ s\ (3.s.f.)}$$

(d)

$$s = \int_0^{-\frac{1}{2}\ln\left(\frac{3}{10}\right)} 10e^{-2t} - 3\ dt$$
$$= \left[-\frac{10}{2}e^{-2t} - 3t\right]_0^{-\frac{1}{2}\ln\left(\frac{3}{10}\right)}$$
$$= 1.694040...$$
$$= \mathbf{1.69\ m\ (3.s.f.)}$$

(e) Note that $10e^{-2t} > 0$

$$\therefore v > -3 \textbf{ (shown)}$$

3. (a) At A, $v = 0$,

$$\begin{aligned} 2e^{0.1t} - 6e^{0.1-0.4t} &= 0 \\ e^{0.1t} &= 3e^{0.1-0.4t} \\ e^{0.1t-(0.1-0.4t)} &= 3 \\ e^{0.5t-0.1} &= 3 \\ \therefore \frac{1}{2}t - \frac{1}{10} &= \ln 3 \end{aligned}$$

$$t = 2\ln 3 + \frac{1}{5} \textbf{ (shown)}$$

□

(b)

$$v = 2e^{0.1t} - 6e^{0.1-0.4t}$$

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= 0.2e^{0.1t} + 2.4e^{0.1-0.4t} \end{aligned}$$

Hence, when $t = 2\ln 3 + \frac{1}{5}$

$$\begin{aligned} a &= 0.2e^{0.1\left(2\ln 3+\frac{1}{5}\right)} + 2.4e^{0.1-0.4\left(2\ln 3+\frac{1}{5}\right)} \\ &= 1.270896... \\ &= \mathbf{1.27\ m/s^2\ (3.s.f.)} \end{aligned}$$

(c)

$$v = 2e^{0.1t} - 6e^{0.1-0.4t}$$

$$\begin{aligned} s &= \int 2e^{0.1t} - 6e^{0.1-0.4t}\ dt \\ &= 20e^{0.1t} + 15e^{0.1-0.4t} + c \end{aligned}$$

When $t = 0$, $s = 0$,

$$\begin{aligned} 0 &= 20e^{0.1(0)} + 15e^{0.1-0.4(0)} + c \\ c &= -\left(20 + 15e^{0.1}\right) \end{aligned}$$

$$\therefore s = 20e^{0.1t} + 15e^{0.1-0.4t} - \left(20 + 15e^{0.1}\right)$$

Hence, when $t = 2\ln 3 + \frac{1}{5}$,

$$\begin{aligned} s &= 20e^{0.1\left(2\ln 3+\frac{1}{5}\right)} + 15e^{0.1-0.4\left(2\ln 3+\frac{1}{5}\right)} - \left(20 + 15e^{0.1}\right) \\ &= 4.805154... \\ &= \mathbf{4.81\ m\ (3.s.f.)} \end{aligned}$$

(d) When $t = 5$,

$$\begin{aligned} s &= 20e^{0.1(5)} + 15e^{0.1-0.4(5)} - \left(20 + 15e^{0.1}\right) \\ &= -1.36 \text{ m} \end{aligned}$$

When $t = 6$,

$$\begin{aligned} s &= 20e^{0.1(6)} + 15e^{0.1-0.4(6)} - \left(20 + 15e^{0.1}\right) \\ &= 1.37 \text{ m} \end{aligned}$$

Since the displacement changes from negative to positive, it passes through O during the 6th second

4. (a) At instantaneous rest, $v = 0$,

$$2t^2 - 8t + 6 = 0$$
$$2(t-1)(t-3) = 0$$
$$\therefore t = \mathbf{1} \qquad \text{or} \qquad t = \mathbf{3}$$

(b)

$$v = 2t^2 - 8t + 6$$
$$a = \frac{dv}{dt}$$
$$= 4t - 8$$

At minimum velocity, $\frac{dv}{dt} = 0$

$$4t - 8 = 0$$
$$t = 2$$

$$\therefore \text{Minimum velocity} = 2(2)^2 - 8(2) + 6$$
$$= \mathbf{-2\ m/s}$$

$$\therefore \text{Particle is moving in the } \textbf{opposite direction}$$

(c)

$$v = 2t^2 - 8t + 6$$
$$s = \int 2t^2 - 8t + 6\ dt$$
$$= 2\left(\frac{t^3}{3}\right) - 8\left(\frac{t^2}{2}\right) + 6t + c$$

At $t = 2$, $s = 1$,

$$1 = 2\left(\frac{8}{3}\right) - 8\left(\frac{4}{2}\right) + 6(2) + c$$
$$c = -\frac{1}{3}$$
$$\therefore s = -\frac{2}{3}t^3 - 4t^2 + 6t - \frac{1}{3}$$

When $t = 0$,

$$s = -\frac{1}{3}$$

When $t = 1$,

$$s = 2\frac{1}{3}$$

When $t = 2$,

$$s = -\frac{1}{3}$$

When $t = 5$,

$$s = 13$$

$$\text{Average speed} = \frac{\frac{1}{3} + \left(2\frac{1}{3} \times 2\right) + \left(\frac{1}{3} \times 2\right) + 13}{5}$$
$$= \mathbf{3\frac{11}{15}\ m/s}$$