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# Sec 3 A-Math Mock Exam (MYE) 2022 (Marking Scheme)

1. (a)

$$y = -2x - 2 \text{ ......(1)}$$

$$\frac{1}{x} + \frac{2}{y} = \frac{1}{2}$$
$$2y + 4x = xy \text{ ......(2) ...... } \textbf{M1}$$

Substitute Equation (1) into Equation (2),

$$2(-2x-2) + 4x = x(-2x-2)$$
$$2x^2 + 2x - 4 = 0 \text{ ...... } \textbf{M1}$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0 \text{ ...... } \textbf{M1}$$

$$\therefore x = -2 \quad \text{or} \quad x = 1$$
$$y = 2 \quad \text{or} \quad y = -4$$

$$\boldsymbol{x = -2,\ y = 2} \qquad \text{and} \qquad \boldsymbol{x = 1,\ y = -4} \text{ ...... } \textbf{A2}$$

(b) (i) Since the curve lies entirely below or above the $x$-axis, $b^2 - 4ac < -0$ ...... **M1**

$$(2k)^2 - 4(k-2)(k+3) < 0 \text{ ...... } \textbf{M1}$$
$$4k^2 - 4k^2 - 4k + 24 < 0$$
$$4k < -24$$
$$k > 6$$

$$\therefore \boldsymbol{k > 6} \text{ ...... } \textbf{A1}$$

(ii) Since $k > 6$, the coefficient of $x^2$ is $(k-2)$ and this is strictly positive ...... **M1**
Hence, the curve lies entirely **above the $x$-axis** ...... **A1**

**Marking comments:**

- **1 mark is awarded if**
  - **Only state coefficient of $x^2 > 0$ with no reference to $k$**
  - **Sub appropriate $k$ value to explain $x^2 > 0$**
  - **Only state $k > 6$ without any reference to $k - 2$**

- **0 mark is awarded if**
  - **No justification**
  - **Sub appropriate $k$ but did not explain $x^2 > 0$**

2. (a)

$$f(x) = x^3 + ax^2 + bx + 3$$

Since $f(x)$ is exactly divisible by $(x+1)$,

$$\begin{aligned} f(-1) &= 0 \\ -1 + a - b + 3 &= 0 \\ a - b &= -2 \text{ ......(1)} \end{aligned}$$ **...... M1**

Since $f(x)$ leaves a remainder of 48 when divided by $(x-3)$,

$$\begin{aligned} f(3) &= 0 \\ 27 + 9a + 3b + 3 &= 48 \\ 9a + 3b &= 18 \text{ ......(2)} \end{aligned}$$ **...... M1**

Solving both Equation (1) and (2)

$$\therefore \boldsymbol{a = 1} \qquad \text{and} \qquad \boldsymbol{b = 3}$$ **...... A2**

**Marking comments:**

- **1 mark is awarded if factor theorem $f(-1) = 0$ and remainder theorem $f(3) = 48$ is demonstrated accurately. Only award this mark if there are major calculation errors**

(b) By long division,

$$\begin{aligned} x^3 + x^2 + 3x + 3 &= 0 \\ (x+1)\left(x^2+3\right) &= 0 \\ x = -1 \quad &\text{or} \quad x^2 + 3 = 0 \end{aligned}$$ **...... M1**

For the quadratic factor,

$$\begin{aligned} x^2 + 3 &= 0 \\ b^2 - 4ac &= (0)^2 - 4(1)(3) \\ &= -12 < 0 \end{aligned}$$

Since the discriminant is less than 0, real roots for the quadratic factor **...... M1**

Hence, there is only **1 real root** $x = -1$ **...... A1**

**Marking comments:**

- **M1 for Long Division**
- **M1 for using discriminant to determine number of roots, or other appropriate methods**

3. (a)

$$\frac{3\left(x^2-2x+9\right)}{(x-3)\left(x^2+9\right)}=\frac{A}{x-3}+\frac{Bx+C}{x^2+9}$$

$$3\left(x^2-2x+9\right)=A\left(x^2+9\right)+(Bx+C)(x-3) \text{ ...... } \mathbf{M1}$$

When $x=3$,

$$3\left[(3)^2-2(3)+9\right]=A\left[(3)^2+9\right]$$
$$A=2 \text{ ...... } \mathbf{M1}$$

When $x=0$,

$$3\left[(0)^2-2(0)+9\right]=2\left[(0)^2+9\right]+C\left[(0)-3\right]$$
$$C=-3 \text{ ...... } \mathbf{M1}$$

When $x=1$,

$$3\left[(1)^2-2(1)+9\right]=2\left[(1)^2+9\right]+(B-3)\left[(1)-3\right]$$
$$B=1 \text{ ...... } \mathbf{M1}$$

$$\therefore \frac{3\left(x^2-2x+9\right)}{(x-3)\left(x^2+9\right)}=\boldsymbol{\frac{2}{x-3}+\frac{x-3}{x^2+9}} \text{ ...... } \mathbf{A1}$$

**Marking comments:**

- **No A1 if there is no final answer**

(b)

$$\left(2+5\sqrt{5}\right)=\frac{1}{3}\text{(base area)}\left(3+2\sqrt{5}\right) \text{ ...... } \mathbf{M1}$$

$$\text{base area}=3\left(\frac{2+5\sqrt{5}}{3+2\sqrt{5}}\right) \text{ ...... } \mathbf{M1}$$

$$=3\left(\frac{2+5\sqrt{5}}{3+2\sqrt{5}}\right)\left(\frac{3-2\sqrt{5}}{3-2\sqrt{5}}\right) \text{ ...... } \mathbf{M1}$$

$$=3\left(\frac{11\sqrt{5}-44}{-11}\right) \text{ ...... } \mathbf{M1}$$

$$=\mathbf{3\left(4-\sqrt{5}\right)\ cm^2} \text{ ...... } \mathbf{A1}$$

4. (a)

$$T_{r+1} = \binom{8}{r} x^{8-r} \left(-\frac{1}{3x}\right)^r$$
$$= \binom{8}{r} \left(-\frac{1}{3}\right)^r x^{8-2r} \text{ ...... } \textbf{M1}$$

Since we are looking for the $x^2$ term,

$$8 - 2r = 2$$
$$r = 3 \text{ ...... } \textbf{M1}$$

$$\therefore \text{Coefficient} = \binom{8}{3} \left(-\frac{1}{3}\right)^3 \text{ ...... } \textbf{M1}$$
$$= \mathbf{-2\frac{2}{27}} \text{ ...... } \textbf{A1}$$

(b) To solve this, we need to split up the binomials and compute them separately

$$(1+3x)^7 = \binom{7}{0}(1)^7(3x)^0 + \binom{7}{1}(1)^6(3x)^1 + \binom{7}{2}(1)^5(3x)^2 + \ldots \text{ ...... } \textbf{M1}$$
$$= 1 + 21x + 189x^2 + \ldots \text{ ...... } \textbf{M1}$$

$$(2-x)^4 = \binom{4}{0}(2)^4(-x)^0 + \binom{4}{1}(2)^3(-x)^1 + \binom{4}{2}(2)^2(-x)^2 + \ldots \text{ ...... } \textbf{M1}$$
$$= 16 - 32x + 24x^2 + \ldots \text{ ...... } \textbf{M1}$$

Hence,

$$(1+3x)^7(2-x)^4 = \left(1 + 21x + 189x^2 + \ldots\right)\left(16 - 32x + 24x^2 + \ldots\right)$$
$$= \ldots + 21x(-32x) + 16\left(189x^2\right) + 24x^2(1) + \ldots \text{ ...... } \textbf{M1}$$
$$= \ldots + 2376x^2 + \ldots$$

$$\therefore \text{Coefficient of } x^2 = \mathbf{2376} \text{ ...... } \textbf{A1}$$

5. (a)

$$\text{Height of prism} = \frac{3\left(x^2-5\right)}{x-1} \text{ ...... M1}$$

Since the height is greater than 10 mm

$$\frac{3\left(x^2-5\right)}{x-1} > 1 \text{ ...... M1}$$

$$3x^2 - 15 > x - 1$$

$$3x^2 - x - 14 > 0 \text{ ...... M1}$$

$$(3x-7)(x+2) > 0 \text{ ...... M1}$$

$$\therefore \boldsymbol{x < -2} \textbf{ (rej.)} \quad \text{or} \quad \boldsymbol{x > 2\frac{1}{3}} \text{ ...... A1}$$

**Marking comments:**

- **No A1 if there is no graph drawn**
- **No A1 if $x < -2$ is not rejected**

(b)

$$\frac{8^x}{5^x} = \frac{5^{3-x}}{27^x}$$

$$8^x \times 27^x = 5^{3-x} \times 5^x \text{ ...... M1}$$

$$216^x = 5^3 \text{ ...... M2}$$

$$6^{3x} = 5^3 \text{ ...... M1}$$

$$6^x = \mathbf{5} \text{ ...... A1}$$

6. (a)

$$(x-a)(b-x) = m$$
$$xb - x^2 - ab + ax - m = 0$$
$$-x^2 + (a+b)x - ab - m = 0$$
$$x^2 - (a+b)x + (ab+m) = 0 \text{ ...... } \textbf{M1}$$

Since the roots are equal, $b^2 - 4ac = 0$ ...... **M1**

$$(a+b)^2 - 4(1)(ab+m) = 0$$
$$a^2 + 2ab + b^2 - 4ab - 4m = 0$$
$$a^2 - 2ab + b^2 - 4m = 0$$
$$(a-b)^2 - 4m = 0 \text{ ...... } \textbf{M1}$$

$$\boldsymbol{m = \left(\frac{a-b}{2}\right)^2} \textbf{ (shown)} \text{ ...... } \textbf{A1}$$

□

(b) (i) We first solve for $\left(1-\sqrt{a}\right)^5$,

$$\begin{aligned}
\left(1-\sqrt{a}\right)^2 &= 1-2\sqrt{a}+a \\
\left(1-\sqrt{a}\right)^4 &= \left(1-2\sqrt{a}+a\right)^2 \\
&= 1-2\sqrt{a}+a-2\sqrt{a}+4a-2a\sqrt{a}+a-2a\sqrt{a}+a^2 \\
&= 1-4\sqrt{a}-4a\sqrt{a}+6a+a^2 \\
\therefore \left(1-\sqrt{a}\right)^5 &= \left(1-4\sqrt{a}-4a\sqrt{a}+6a+a^2\right)\left(1-\sqrt{a}\right) \\
&= 1-\sqrt{a}-4\sqrt{a}+4a-4a\sqrt{a}+4a^2+6a-6a\sqrt{a}+a^2-a^2\sqrt{a} \\
&= 1-5\sqrt{a}-10a\sqrt{a}-a^2\sqrt{a}+10a+5a^2 \text{ ...... } \textbf{M1}
\end{aligned}$$

Next, for $\left(1+\sqrt{a}\right)^5$, by inspection,

$$\left(1+\sqrt{a}\right)^5 = 1+5\sqrt{a}+10a\sqrt{a}+a^2\sqrt{a}+10a+5a^2 \text{ ...... } \textbf{M1}$$

$$\begin{aligned}
&\therefore \left(1-\sqrt{a}\right)^5-\left(1+\sqrt{a}\right)^5 \\
&= \left[1-5\sqrt{a}-10a\sqrt{a}-a^2\sqrt{a}+10a+5a^2\right]-\left[1+5\sqrt{a}+10a\sqrt{a}+a^2\sqrt{a}+10a+5a^2\right] \\
&= 1-5\sqrt{a}-10a\sqrt{a}-a^2\sqrt{a}+10a+5a^2-1-5\sqrt{a}-10a\sqrt{a}-a^2\sqrt{a}-10a-5a^2 \\
&= \boldsymbol{-10\sqrt{a}-20a\sqrt{a}-2a^2\sqrt{a}} \textbf{ (shown)} \text{ ...... } \textbf{A2}
\end{aligned}$$

(ii) By comparing part (a) and (b),

$$a = 3 \text{ ...... } \textbf{M1}$$

$$\begin{aligned}
\therefore \left(1-\sqrt{3}\right)^5-\left(1+\sqrt{3}\right)^5 &= -10\sqrt{3}-20(3)\sqrt{3}-2(3)^2\sqrt{3} \\
&= \boldsymbol{-88\sqrt{3}} \text{ ...... } \textbf{A1}
\end{aligned}$$

**Alternative method for part (a)**

The intial part of the question can also be solved using the Binomial Theorem

$$\begin{aligned}
&\left(1-\sqrt{a}\right)^5 \\
&= 1+\binom{5}{1}\left(-\sqrt{a}\right)+\binom{5}{2}\left(-\sqrt{a}\right)^2+\binom{5}{3}\left(-\sqrt{a}\right)^3+\binom{5}{4}\left(-\sqrt{a}\right)^4+\left(-\sqrt{a}\right)^5 \\
&= 1-5\sqrt{a}+10a-10a^{1\frac{1}{2}}+5a^2-a^{2\frac{1}{2}} \\
&= 1-5a-10a\sqrt{a}-a^2\sqrt{a}+10a+5a^2
\end{aligned}$$

The remaining part of the question is the same

7. **Bonus Question**

Since $\{x\} + \{x^2\} = 1$, the value $x$ must satisfy $x + x^2 = n$ for some integer $n$. Hence, the solution of $x$ is given as

$$x = \frac{-1 \pm \sqrt{1+4n}}{2}$$

If we consider, when $0 \leq x \leq 8$, we must have

$$\frac{-1 + \sqrt{1+4n}}{2} \leq 8$$

Solving this inequality, we find that $x$ satisfies the equation when $0 \leq n \leq 72$, giving us 73 possibilities

Likewise, when $-8 \leq x \leq 0$, we must have

$$\frac{-1 - \sqrt{1+4n}}{2} \geq -8$$

Solving this inequality, we find that $x$ satisfies the equation when $0 \leq n \leq 56$, giving us 57 possibilities

Since $\{x\} + \{x^2\} < 2$, we must eliminate cases when $\{x\} + \{x^2\} = 0$, which happens when $-8 \leq x \leq 8$ is an integer, for a total of 17 possibilities

$$\text{Total possibilities} = 73 + 57 - 17 = \mathbf{113\ (shown)}$$

□

**Can refer to the solutions posted on the SMT (Stanford Math Tournament) website, 2018 Algebra Paper Qn 8. Solutions are the same as the solutions presented above**

**Examiners' Comments:**

- More difficult questions: Qn 4(b), 5(b), 6(b)
- Not a difficult paper, except maybe the 3 questions listed above, that may pose some difficulties for some students