A LEVEL
H2 MATHEMATICS
INTEGRATION TECHNIQUES

CHAPTER ANALYSIS

MASTERY

- Basic Integration
- Standard Integration Rules
- Integration by Substitution
- Integration by Parts

EXAM

- Need to know for subsequent chapter, such as applications of integration and differential equations
- Usually tested within these other chapters

WEIGHTAGE

- Rarely appears on its own, you're expected to integrate usually in contextual/application questions
- That being said, considered high weightage

Recap: Basic Integration

	Basic Rules of Integration
1	$\int [f(x) \pm g(x)]\, dx = \int f(x)dx \pm \int g(x)dx$
2	$\int kf(x)\, dx = k \int f(x)\, dx$ where k is a constant, $k \neq 0$
3	$\int \frac{d}{dx} f(x)\, dx = f(x) + c$
4	$\frac{d}{dx} \int f(x)\, dx = f(x)$
5	Note: $\int [f(x)g(x)]\, dx \neq \int f(x)\, dx \cdot \int g(x)\, dx$ $\int \frac{f(x)}{g(x)} dx \neq \frac{\int f(x)\, dx}{\int g(x)\, dx}$

$f'(x)$	$f(x)$
x^n	$\frac{x^{n+1}}{n+1} + c\,, n \neq 1$
$f'(x)[f(x)]^n$	$\frac{[f(x)]^{n+1}}{n+1} + c\,, n \neq 1$
e^x	$e^x + c$
$f'(x)e^{f(x)}$	$e^{f(x)} + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
$\frac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert + c$

$f'(x)$	$f(x)$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\sec^2 x$	$\tan x + c$
$f'(x)\cos f(x)$	$\sin f(x) + c$
$f'(x)\sin f(x)$	$-\cos f(x) + c$
$f'(x)\sec^2 f(x)$	$\tan f(x) + c$

Practice

$$\int \frac{x}{(3x^2+1)} dx$$
$$= \frac{1}{6} \int \frac{6x}{(3x^2+1)} dx$$
$$= \frac{1}{6} \ln|3x^2+1| + c$$

Practice

$$\int 3^{2x+1}\, dx = \int e^{(2x+1)\ln 3}\, dx$$
$$= \frac{1}{2\ln 3} \int (2\ln 3)\, e^{(2x+1)\ln 3}\, dx$$
$$= \frac{1}{2\ln 3}\, e^{(2x+1)\ln 3} + c$$
$$= \frac{1}{2\ln 3}\, 3^{2x+1} + c$$

Practice

$$\int \frac{x}{(3x^2+1)} dx$$
$$= \frac{1}{6} \int \frac{6x}{(3x^2+1)} dx$$
$$= \frac{1}{6} \ln|3x^2+1| + c$$

Practice

$$\int x \sin(5x^2)\, dx$$
$$= \frac{1}{10} \int 10x \sin(5x^2)\, dx$$
$$= -\frac{1}{10} x \cos(5x^2) + c$$

Other Trigonometric Integration Techniques

From MF26

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A \equiv 2 \sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin P + \sin Q \equiv 2 \sin \tfrac{1}{2}(P+Q) \cos \tfrac{1}{2}(P-Q)$$

$$\sin P - \sin Q \equiv 2 \cos \tfrac{1}{2}(P+Q) \sin \tfrac{1}{2}(P-Q)$$

$$\cos P + \cos Q \equiv 2 \cos \tfrac{1}{2}(P+Q) \cos \tfrac{1}{2}(P-Q)$$

$$\cos P - \cos Q \equiv -2 \sin \tfrac{1}{2}(P+Q) \sin \tfrac{1}{2}(P-Q)$$

Useful Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\cot^2 x + 1 = \operatorname{cosec}^2 x$$

From Factor Formula, Derive:

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2 \sin\alpha \cos\beta$$
$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2 \cos\alpha \sin\beta$$
$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2 \cos\alpha \cos\beta$$
$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2 \sin\alpha \sin\beta$$

Not found in MF26, but useful!

Practice

$$\int \sin x \cos 3x \, dx$$

$$= \frac{1}{2} \int 2 \sin 3x \cos x \, dx$$

$$= \frac{1}{2} \int \sin(3x + x) + \sin(3x - x) \, dx$$

$$= \frac{1}{2} \int \sin 4x + \sin 2x \, dx$$

$$= \frac{1}{2}\left[-\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x\right] + c$$

$$= -\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + c$$

Standard Integration Rules

From MF26

f(x)	$\int f(x)\,dx$	
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$	$(\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$	$(x > a)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$	$(\lvert x\rvert < a)$
$\tan x$	$\ln(\sec x)$	$(\lvert x\rvert < \frac{1}{2}\pi)$
$\cot x$	$\ln(\sin x)$	$(0 < x < \pi)$
$\operatorname{cosec} x$	$-\ln(\operatorname{cosec} x + \cot x)$	$(0 < x < \pi)$
$\sec x$	$\ln(\sec x + \tan x)$	$(\lvert x\rvert < \frac{1}{2}\pi)$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

Practice

$$\int \frac{3}{5+x^2}\,dx = \frac{3}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

Practice

$$\int \frac{x}{\sqrt{4-x^6}}\,dx = \frac{1}{3}\int \frac{3x}{\sqrt{2^2-(x^3)^2}}\,dx$$

$$= \frac{1}{3}\sin^{-1}\frac{x^3}{2} + c$$

Practice

$$\int \frac{1}{x^2+6x-7}\,dx$$

$$= \int \frac{1}{x^2+6x+3^2-3^2-7}\,dx$$

$$= \int \frac{1}{(x+3)^2-4^2}\,dx$$

$$= \frac{1}{2(4)}\ln\left|\frac{x+3-4}{x+3+4}\right| + c$$

$$= \frac{1}{8}\ln\left|\frac{x-1}{x+7}\right| + c$$

Integration By Substitution

Example: Use the substitution $\frac{1}{2}x = \tan u$ to find $\int \frac{1}{\sqrt{4+x^2}}dx$

Start with the substitution given in the question

$$\frac{1}{2}x = \tan u$$

$$x = 2\tan u \text{ --- (1)}$$

Differentiate wrt u

$$\frac{dx}{du} = 2\sec^2 u$$

$$dx = 2\sec^2 u \;\; du \text{ ---(2)}$$

Sub (1) and (2) into $\int \frac{1}{4+x^2}dx$

$$\int \frac{1}{\sqrt{4+x^2}}dx = \int \frac{1}{\sqrt{4+(2\tan u)^2}} \times \; 2\sec^2 u\, du$$

$$= \int \frac{1}{\sqrt{4+4\tan^2 u}} \times \; 2\sec^2 u\, du = \int \frac{1}{2\sqrt{1+\tan^2 u}} \times \; 2\sec^2 u\, du$$

$$= \int \frac{1}{2\sqrt{\sec^2 u}} \times \; 2\sec^2 u\, du \; = \int \frac{2\sec^2 u}{2\sec u}\, du$$

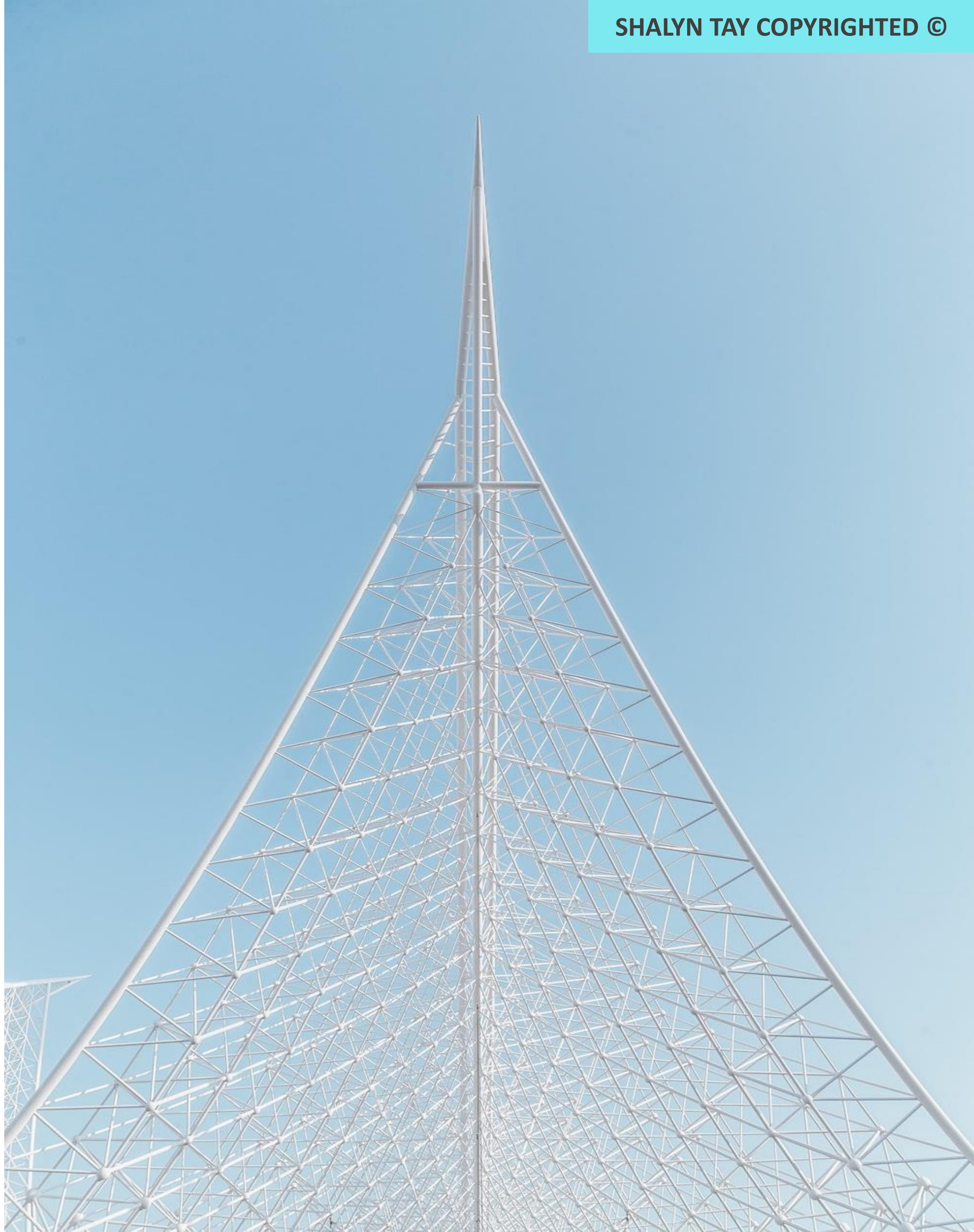
$$= \int \sec u \;\; du = \ln|\sec u + \tan u| + c$$

Sub *x* back in

$$\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + c = \ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| + c$$

General Formula

$$\int f(x)dx = \int f(x(u))\frac{dx}{du}du$$



Integration By Parts

General Formula

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Order Of Choice For u

LIATE

Logarithm
Inverse Trigo
Algebraic
Trigo
Exponential

Example: Find $\int x^3\, e^{3x} dx$

Use LIATE to determine:

$u = x^3$	$\frac{dv}{dx} = e^{3x}$
$\frac{du}{dx} = 3x$	$v = \frac{e^{3x}}{3}$

$$\int x^3\, e^{3x} dx = (x^3)\left(\frac{e^{3x}}{3}\right) - \int \left(\frac{e^{3x}}{3}\right)(3x)dx$$

$$= \frac{1}{2}x^3 e^{3x} - \boxed{\int x e^{3x} dx}$$

Integrate by parts again

$u = x$	$\frac{dv}{dx} = e^{3x}$
$\frac{du}{dx} = 1$	$v = \frac{e^{3x}}{3}$

$$\frac{1}{2}x^3 e^{3x} - \int x e^{3x} dx = \frac{1}{2}x^3 e^{3x} - \left[(x)\left(\frac{e^{3x}}{3}\right) - \int \left(\frac{e^{3x}}{3}\right)(1)dx\right]$$

$$= \frac{1}{2}x^3 e^{3x} - \frac{1}{3}x e^{3x} + \frac{1}{3}\int (e^{3x})dx$$

$$= \frac{1}{2}x^3 e^{3x} - \frac{1}{3}x e^{3x} + \frac{1}{9}e^{3x} + c$$