

A LEVEL
H2 MATHEMATICS
INTEGRATION TECHNIQUES

CHAPTER ANALYSIS



MASTERY

- Basic Integration
- Standard Integration Rules
- Integration by Substitution
- Integration by Parts



EXAM

- Need to know for subsequent chapter, such as applications of integration and differential equations
- Usually tested within these other chapters



WEIGHTAGE

- Rarely appears on its own, you're expected to integrate usually in contextual/application questions
- That being said, considered high weightage

Recap: Basic Integration

Basic Rules of Integration	
1	$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$
2	$\int kf(x) dx = k \int f(x) dx$ where k is a constant, $k \neq 0$
3	$\int \frac{d}{dx} f(x) dx = f(x) + c$
4	$\frac{d}{dx} \int f(x) dx = f(x)$
5	Note: $\int [f(x)g(x)] dx \neq \int f(x) dx \cdot \int g(x) dx$ $\int \frac{f(x)}{g(x)} dx \neq \frac{\int f(x) dx}{\int g(x) dx}$

$f'(x)$	$f(x)$
x^n	$\frac{x^{n+1}}{n+1} + c, n \neq -1$
$f'(x)[f(x)]^n$	$\frac{[f(x)]^{n+1}}{n+1} + c, n \neq -1$
e^x	$e^x + c$
$f'(x)e^{f(x)}$	$e^{f(x)} + c$
$\frac{1}{x}$	$\ln x + c$
$\frac{f'(x)}{f(x)}$	$\ln f(x) + c$

$f'(x)$	$f(x)$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\sec^2 x$	$\tan x + c$
$f'(x) \cos f(x)$	$\sin f(x) + c$
$f'(x) \sin f(x)$	$-\cos f(x) + c$
$f'(x) \sec^2 f(x)$	$\tan f(x) + c$

Practice

$$\begin{aligned} & \int \frac{x}{(3x^2 + 1)} dx \\ &= \frac{1}{6} \int \frac{6x}{(3x^2 + 1)} dx \\ &= \frac{1}{6} \ln|3x^2 + 1| + c \end{aligned}$$

Practice

$$\begin{aligned} & \int 3^{2x+1} dx = \int e^{(2x+1) \ln 3} dx \\ &= \frac{1}{2 \ln 3} \int (2 \ln 3) e^{(2x+1) \ln 3} dx \\ &= \frac{1}{2 \ln 3} e^{(2x+1) \ln 3} + c \\ &= \frac{1}{2 \ln 3} 3^{2x+1} + c \end{aligned}$$

Practice

$$\begin{aligned} & \int \frac{x}{(3x^2 + 1)} dx \\ &= \frac{1}{6} \int \frac{6x}{(3x^2 + 1)} dx \\ &= \frac{1}{6} \ln|3x^2 + 1| + c \end{aligned}$$

Practice

$$\begin{aligned} & \int x \sin(5x^2) dx \\ &= \frac{1}{10} \int 10x \sin(5x^2) dx \\ &= -\frac{1}{10} x \cos(5x^2) + c \end{aligned}$$

Other Trigonometric Integration Techniques

From MF26

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A \equiv 2 \sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A$$

$$\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin P + \sin Q \equiv 2 \sin \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)$$

$$\sin P - \sin Q \equiv 2 \cos \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)$$

$$\cos P + \cos Q \equiv 2 \cos \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)$$

$$\cos P - \cos Q \equiv -2 \sin \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)$$

Useful Trigonometric Identities

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \tan^2 x + 1 &= \sec^2 x \\ \cot^2 x + 1 &= \operatorname{cosec}^2 x \end{aligned}$$

Practice

$$\begin{aligned} &\int \sin x \cos 3x \, dx \\ &= \frac{1}{2} \int 2 \sin 3x \cos x \, dx \\ &= \frac{1}{2} \int \sin(3x + x) + \sin(3x - x) \, dx \\ &= \frac{1}{2} \int \sin 4x + \sin 2x \, dx \\ &= \frac{1}{2} \left[-\frac{1}{4} \cos 4x - \frac{1}{2} \cos 2x \right] + c \\ &= -\frac{1}{8} \cos 4x - \frac{1}{4} \cos 2x + c \end{aligned}$$

From Factor Formula, Derive:

$$\begin{aligned} \sin(\alpha + \beta) + \sin(\alpha - \beta) &= 2 \sin \alpha \cos \beta \\ \sin(\alpha + \beta) - \sin(\alpha - \beta) &= 2 \cos \alpha \sin \beta \\ \cos(\alpha + \beta) + \cos(\alpha - \beta) &= 2 \cos \alpha \cos \beta \\ \cos(\alpha + \beta) - \cos(\alpha - \beta) &= -2 \sin \alpha \sin \beta \end{aligned}$$

Not found in MF26, but useful!

Standard Integration Rules

$f(x)$	$\int f(x) dx$	
$\frac{1}{x^2 + a^2}$	$\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$	
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$	$(x < a)$
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right)$	$(x > a)$
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right)$	$(x < a)$
$\tan x$	$\ln(\sec x)$	$(x < \frac{1}{2}\pi)$
$\cot x$	$\ln(\sin x)$	$(0 < x < \pi)$
$\operatorname{cosec} x$	$-\ln(\operatorname{cosec} x + \cot x)$	$(0 < x < \pi)$
$\sec x$	$\ln(\sec x + \tan x)$	$(x < \frac{1}{2}\pi)$

From MF26

$$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a} \tan^{-1} \frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1} \frac{f(x)}{a} + c$$

Practice

$$\int \frac{1}{x^2 + 6x - 7} dx$$

$$= \int \frac{1}{x^2 + 6x + 3^2 - 3^2 - 7} dx$$

$$= \int \frac{1}{(x + 3)^2 - 4^2} dx$$

$$= \frac{1}{2(4)} \ln \left| \frac{x + 3 - 4}{x + 3 + 4} \right| + c$$

$$= \frac{1}{8} \ln \left| \frac{x - 1}{x + 7} \right| + c$$

Practice

$$\int \frac{3}{5 + x^2} dx = \frac{3}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}} + c$$

Practice

$$\int \frac{x}{\sqrt{4 - x^6}} dx = \frac{1}{3} \int \frac{3x}{\sqrt{2^2 - (x^3)^2}} dx$$

$$= \frac{1}{3} \sin^{-1} \frac{x^3}{2} + c$$

Integration By Substitution

Example: Use the substitution $\frac{1}{2}x = \tan u$ to find $\int \frac{1}{\sqrt{4+x^2}} dx$

Start with the substitution given in the question

$$\frac{1}{2}x = \tan u$$

$$x = 2 \tan u \quad \text{--- (1)}$$

Differentiate wrt u

$$\frac{dx}{du} = 2 \sec^2 u$$

$$dx = 2 \sec^2 u \, du \quad \text{---(2)}$$

General Formula

$$\int f(x) dx = \int f(x(u)) \frac{dx}{du} du$$

Sub (1) and (2) into $\int \frac{1}{4+x^2} dx$

$$\int \frac{1}{\sqrt{4+x^2}} dx = \int \frac{1}{\sqrt{4+(2 \tan u)^2}} \times 2 \sec^2 u \, du$$

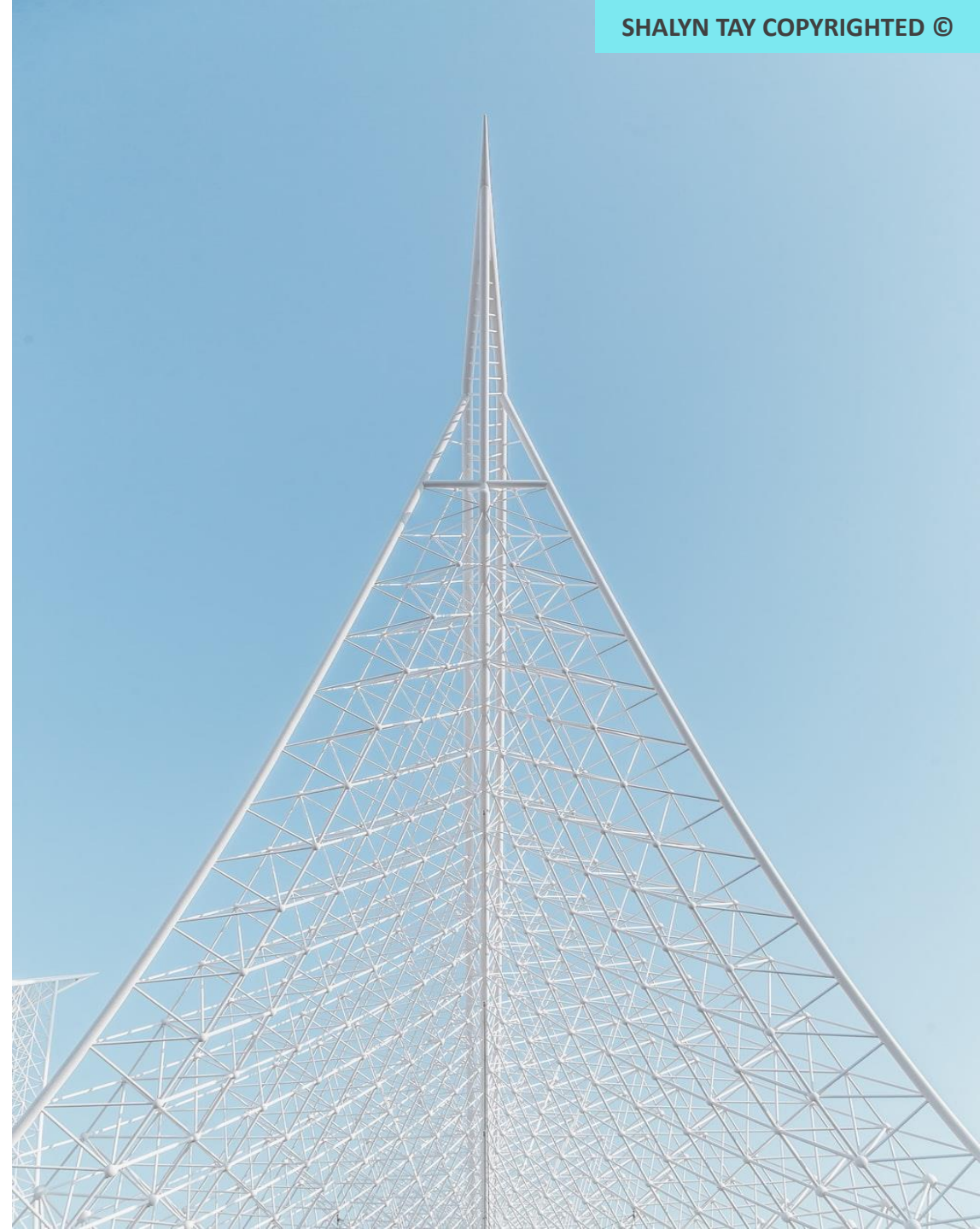
$$= \int \frac{1}{\sqrt{4+4 \tan^2 u}} \times 2 \sec^2 u \, du = \int \frac{1}{2\sqrt{1+\tan^2 u}} \times 2 \sec^2 u \, du$$

$$= \int \frac{1}{2\sqrt{\sec^2 u}} \times 2 \sec^2 u \, du = \int \frac{2 \sec^2 u}{2 \sec u} \, du$$

$$= \int \sec u \, du = \ln|\sec u + \tan u| + c$$

Sub x back in

$$\ln \left| \frac{\sqrt{x^2+4}}{2} + \frac{x}{2} \right| + c = \ln \left| \frac{\sqrt{x^2+4} + x}{2} \right| + c$$



Integration By Parts

General Formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Order Of Choice For u

LIATE

Logarithm
Inverse Trigo
Algebraic
Trigo
Exponential

Example: Find $\int x^3 e^{3x} dx$

Use LIATE to determine:

$u = x^3$	$\frac{dv}{dx} = e^{3x}$
$\frac{du}{dx} = 3x$	$v = \frac{e^{3x}}{3}$

$$\int x^3 e^{3x} dx = (x^3) \left(\frac{e^{3x}}{3} \right) - \int \left(\frac{e^{3x}}{3} \right) (3x) dx$$

$$= \frac{1}{2} x^3 e^{3x} - \int x e^{3x} dx$$

Integrate by parts again

$u = x$	$\frac{dv}{dx} = e^{3x}$
$\frac{du}{dx} = 1$	$v = \frac{e^{3x}}{3}$

$$\frac{1}{2} x^3 e^{3x} - \int x e^{3x} dx = \frac{1}{2} x^3 e^{3x} - \left[(x) \left(\frac{e^{3x}}{3} \right) - \int \left(\frac{e^{3x}}{3} \right) (1) dx \right]$$

$$= \frac{1}{2} x^3 e^{3x} - \frac{1}{3} x e^{3x} + \frac{1}{3} \int (e^{3x}) dx$$

$$= \frac{1}{2} x^3 e^{3x} - \frac{1}{3} x e^{3x} + \frac{1}{9} e^{3x} + c$$




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