

# A LEVEL H2 MATHEMATICS INTEGRATION TECHNIQUES

# CHAPTER ANALYSIS



MASTERY



EXAM



WEIGHTAGE

- Basic Integration
  - Standard Integration Rules
  - Integration by Substitution
  - Integration by Parts
- 
- Need to know for subsequent chapter, such as applications of integration and differential equations
  - Usually tested within these other chapters
- 
- Rarely appears on its own, you're expected to integrate usually in contextual/application questions
  - That being said, considered high weightage

## Recap: Basic Integration

<b>Basic Rules of Integration</b>	
1	$\int [f(x) \pm g(x)] dx = \int f(x)dx \pm \int g(x)dx$
2	$\int kf(x) dx = k \int f(x) dx$ where k is a constant, $k \neq 0$
3	$\int \frac{d}{dx} f(x) dx = f(x) + c$
4	$\frac{d}{dx} \int f(x) dx = f(x)$
5	Note: $\int [f(x)g(x)] dx \neq \int f(x) dx \cdot \int g(x) dx$ $\int \frac{f(x)}{g(x)} dx \neq \frac{\int f(x) dx}{\int g(x) dx}$

$f'(x)$	$f(x)$
$x^n$	$\frac{x^{n+1}}{n+1} + c, n \neq 1$
$f'(x)[f(x)]^n$	$\frac{[f(x)]^{n+1}}{n+1} + c, n \neq 1$
$e^x$	$e^x + c$
$f'(x)e^{f(x)}$	$e^{f(x)} + c$
$\frac{1}{x}$	$\ln x  + c$
$\frac{f'(x)}{f(x)}$	$\ln f(x)  + c$

$f'(x)$	$f(x)$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\sec^2 x$	$\tan x + c$
$f'(x) \cos f(x)$	$\sin f(x) + c$
$f'(x) \sin f(x)$	$-\cos f(x) + c$
$f'(x) \sec^2 f(x)$	$\tan f(x) + c$

### Practice

$$\begin{aligned}
 & \int \frac{x}{(3x^2 + 1)} dx \\
 &= \frac{1}{6} \int \frac{6x}{(3x^2 + 1)} dx \\
 &= \frac{1}{6} \ln|3x^2 + 1| + c
 \end{aligned}$$

### Practice

$$\begin{aligned}
 \int 3^{2x+1} dx &= \int e^{(2x+1)\ln 3} dx \\
 &= \frac{1}{2 \ln 3} \int (2 \ln 3) e^{(2x+1)\ln 3} dx \\
 &= \frac{1}{2 \ln 3} e^{(2x+1)\ln 3} + c \\
 &= \frac{1}{2 \ln 3} 3^{2x+1} + c
 \end{aligned}$$

### Practice

$$\begin{aligned}
 & \int \frac{x}{(3x^2 + 1)} dx \\
 &= \frac{1}{6} \int \frac{6x}{(3x^2 + 1)} dx \\
 &= \frac{1}{6} \ln|3x^2 + 1| + c
 \end{aligned}$$

### Practice

$$\begin{aligned}
 & \int x \sin(5x^2) dx \\
 &= \frac{1}{10} \int 10x \sin(5x^2) dx \\
 &= -\frac{1}{10} x \cos(5x^2) + c
 \end{aligned}$$

## Other Trigonometric Integration Techniques

### From MF26

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A \equiv 2 \sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A$$

$$\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin P + \sin Q \equiv 2 \sin \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q)$$

$$\sin P - \sin Q \equiv 2 \cos \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q)$$

$$\cos P + \cos Q \equiv 2 \cos \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q)$$

$$\cos P - \cos Q \equiv -2 \sin \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q)$$

### Useful Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\cot^2 x + 1 = \operatorname{cosec}^2 x$$

### Practice

$$\int \sin x \cos 3x \, dx$$

$$= \frac{1}{2} \int 2 \sin 3x \cos x \, dx$$

$$= \frac{1}{2} \int \sin(3x+x) + \sin(3x-x) \, dx$$

$$= \frac{1}{2} \int \sin 4x + \sin 2x \, dx$$

$$= \frac{1}{2} \left[ -\frac{1}{4} \cos 4x - \frac{1}{2} \cos 2x \right] + c$$

$$= -\frac{1}{8} \cos 4x - \frac{1}{4} \cos 2x + c$$

### From Factor Formula, Derive:

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$$

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta$$

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2 \sin \alpha \sin \beta$$

***Not found in MF26, but useful!***

## Standard Integration Rules

**From MF26**

$$f(x)$$

$$\int f(x) dx$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2} dx = \frac{1}{a} \tan^{-1} \frac{f(x)}{a} + c$$

$$\frac{1}{x^2 + a^2}$$

$$\frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right)$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = \sin^{-1} \frac{f(x)}{a} + c$$

$$\frac{1}{\sqrt{a^2 - x^2}}$$

$$\sin^{-1} \left( \frac{x}{a} \right) \quad (|x| < a)$$

$$\frac{1}{x^2 - a^2}$$

$$\frac{1}{2a} \ln \left( \frac{x-a}{x+a} \right) \quad (x > a)$$

$$\frac{1}{a^2 - x^2}$$

$$\frac{1}{2a} \ln \left( \frac{a+x}{a-x} \right) \quad (|x| < a)$$

$$\tan x$$

$$\ln(\sec x) \quad (|x| < \frac{1}{2}\pi)$$

$$\cot x$$

$$\ln(\sin x) \quad (0 < x < \pi)$$

$$\csc x$$

$$-\ln(\csc x + \cot x) \quad (0 < x < \pi)$$

$$\sec x$$

$$\ln(\sec x + \tan x) \quad (|x| < \frac{1}{2}\pi)$$

**Practice**

$$\begin{aligned} & \int \frac{1}{x^2 + 6x - 7} dx \\ &= \int \frac{1}{x^2 + 6x + 3^2 - 3^2 - 7} dx \\ &= \int \frac{1}{(x+3)^2 - 4^2} dx \\ &= \frac{1}{2(4)} \ln \left| \frac{x+3-4}{x+3+4} \right| + c \\ &= \frac{1}{8} \ln \left| \frac{x-1}{x+7} \right| + c \end{aligned}$$

**Practice**

$$\int \frac{3}{5+x^2} dx = \frac{3}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}} + c$$

**Practice**

$$\begin{aligned} \int \frac{x}{\sqrt{4-x^6}} dx &= \frac{1}{3} \int \frac{3x}{\sqrt{2^2-(x^3)^2}} dx \\ &= \frac{1}{3} \sin^{-1} \frac{x^3}{2} + c \end{aligned}$$

## Integration By Substitution

**Example:** Use the substitution  $\frac{1}{2}x = \tan u$  to find  $\int \frac{1}{\sqrt{4+x^2}} dx$

Start with the substitution given in the question

**General Formula**

$$\int f(x)dx = \int f(x(u)) \frac{dx}{du} du$$

$$\frac{1}{2}x = \tan u$$

$$x = 2 \tan u \quad \text{--- (1)}$$

Differentiate wrt u

$$\frac{dx}{du} = 2\sec^2 u$$

$$dx = 2\sec^2 u \ du \quad \text{---(2)}$$

Sub (1) and (2) into  $\int \frac{1}{\sqrt{4+x^2}} dx$

$$\begin{aligned} \int \frac{1}{\sqrt{4+x^2}} dx &= \int \frac{1}{\sqrt{4 + (2\tan u)^2}} \times 2\sec^2 u du \\ &= \int \frac{1}{\sqrt{4 + 4\tan^2 u}} \times 2\sec^2 u du = \int \frac{1}{2\sqrt{1+\tan^2 u}} \times 2\sec^2 u du \\ &= \int \frac{1}{2\sec^2 u} \times 2\sec^2 u du = \int \frac{2\sec^2 u}{2\sec u} du \\ &= \int \sec u \ du = \ln|\sec u + \tan u| + c \end{aligned}$$

Sub x back in

$$\ln \left| \frac{\sqrt{x^2+4}}{2} + \frac{x}{2} \right| + c = \ln \left| \frac{\sqrt{x^2+4} + x}{2} \right| + c$$



## Integration By Parts



### General Formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

### Order Of Choice For $u$

**LIATE**

Logarithm  
Inverse Trigo  
Algebraic  
Trigo  
Exponential

Use LIATE to determine:

**Example:** Find  $\int x^3 e^{3x} dx$

$u = x^3$	$\frac{dv}{dx} = e^{3x}$
$\frac{du}{dx} = 3x$	$v = \frac{e^{3x}}{3}$

$$\begin{aligned}\int x^3 e^{3x} dx &= (x^3) \left( \frac{e^{3x}}{3} \right) - \int \left( \frac{e^{3x}}{3} \right) (3x) dx \\ &= \frac{1}{2} x^3 e^{3x} - \boxed{\int x e^{3x} dx} \quad \text{Integrate by parts again}\end{aligned}$$

$u = x$	$\frac{dv}{dx} = e^{3x}$
$\frac{du}{dx} = 1$	$v = \frac{e^{3x}}{3}$

$$\begin{aligned}\frac{1}{2} x^3 e^{3x} - \int x e^{3x} dx &= \frac{1}{2} x^3 e^{3x} - \left[ (x) \left( \frac{e^{3x}}{3} \right) - \int \left( \frac{e^{3x}}{3} \right) (1) dx \right] \\ &= \frac{1}{2} x^3 e^{3x} - \frac{1}{3} x e^{3x} + \frac{1}{3} \int (e^{3x}) dx \\ &= \frac{1}{2} x^3 e^{3x} - \frac{1}{3} x e^{3x} + \frac{1}{9} e^{3x} + c\end{aligned}$$



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