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March Practice Questions 2022 Full Solutions (A-Math)

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Question Source

All questions are sourced and selected based on the known abilities of students sitting for the 'O' Level A-Math Examination. All questions compiled here are from **2017-2018 School Mid-Year / Prelim Papers**. Questions are categorised into the various topics and range in varying difficulties. If questions are sourced from respective sources, credit will be given when appropriate.

How to read:

[S4 ABCSS P1/2011 PRELIM Qn 1]

Secondary 4, ABC Secondary School, Paper 1, 2011, Prelim, Question 1

Syllabus (4049)

Algebra	**Geometry and Trigonometry**	**Calculus**
Quadratic Equations & Inequalities	Trigonometry	Differentiation
Surds	Coordinate Geometry	Integration
Polynomials	Further Coordinate Geometry	Kinematics
Simultaneous Equations	Linear Law	
Partial Fractions	Proofs of Plane Geometry	
Binomial Theorem		
Exponential & Logarithms		

Contents

1 Quadratic Equations & Inequalities

1.1 Full Solutions

1. (a) When $k = -20$ and given that $y < 0$,

$$\begin{aligned} 2x^2 - 6x - 20 &< 0 \\ x^2 - 3x - 10 &< 0 \\ (x-5)(x+2) &< 0 \end{aligned}$$

$$\boldsymbol{-2 < x < 5}$$

(b) When $k = 10$,

$$y = 2x^2 - 6x + 10 \text{ ......(1)}$$
$$y + 2x = 8 \text{ ......(2)}$$

Let Equation (1) = Equation (2),

$$\begin{aligned} 2x^2 - 6x + 10 + 2x &= 8 \\ 2x^2 - 4x + 2 &= 0 \\ x^2 - 2x + 1 &= 0 \end{aligned}$$

To show that the line is tangential to the curve, WTS: $b^2 - 4ac = 0$

$$\begin{aligned} \therefore b^2 - 4ac &= (-2)^2 - 4(1)(1) \\ &= 0 \textbf{ (shown)} \end{aligned}$$

□

2. (a) Since the solutions are $\frac{1}{4} < x < 1$ respectively

$$\begin{aligned} (4x-1)(x-1) &= 0 \\ 4x^2 - 5x + 1 &= 0 \\ -4x^2 + 5x - 1 &= 0 \end{aligned}$$

$$\therefore \boldsymbol{a = 4} \qquad \boldsymbol{b = 5}$$

(b) Since the curve lies completely below the line $y = 1 - 4x$,

$$\begin{aligned} -4x^2 + 5x - 1 &< 1 - 4x \\ 4x^2 - 9x + 2 &> 0 \\ (4x-1)(x-2) &> 0 \end{aligned}$$

$$\therefore \boldsymbol{x < \frac{1}{4}} \quad \textbf{or} \quad \boldsymbol{x > 2}$$

3. (a)

$$px^2 + 8x + p > 6$$
$$px^2 + 8x + (p - 6) > 0$$

Since the curve is strictly above the x-axis, $b^2 - 4ac < 0$

$$(8)^2 - 4(p)(p - 6) < 0$$
$$-4p^2 + 24p + 64 < 0$$
$$p^2 - 6p - 16 > 0$$
$$(p + 2)(p - 8) > 0$$

$$\therefore p < -2 \qquad p > 8$$

Since the curve is strictly above the x-axis, $p > 0$

$$\therefore \boldsymbol{p > 8}$$

(b)

$$y + qx = q \text{ ......(1)}$$
$$y = (q + 1)x^2 + qx - 1 \text{ ......(2)}$$

Let Equation (1) = Equation (2),

$$(q + 1)x^2 + qx - 1 + qx = q$$
$$(q + 1)x^2 + 2qx + (-q - 1) = 0$$

To show that the line will intersect the curve at 2 distinct points, WTS: $b^2 - 4ac > 0$

$$b^2 - 4ac = (2q)^2 - 4(q + 1)(-q - 1)$$
$$= 4q^2 + 4(q + 1)^2$$

Since $4q^2 \geq 0$ and $4(q + 1)^2 > 0$

$$\therefore b^2 - 4ac > 0 \textbf{ (shown)}$$

□

4. (a) Since $px^2 + qx + 2q$ is always negative, $b^2 - 4ac < 0$ and $p < 0$

$$(q)^2 - 4(p)(2q) < 0$$
$$q^2 - 8pq < 0$$
$$q(q - 8p) < 0$$

$$\therefore \boldsymbol{p < 0 \quad \& \quad 8p < q < 0}$$

(b) Any value of p and q as long as

- p and q are negative
- $8p < q$

5. (a)

$$y = 2x^2 + 5x + 8 \text{ ......(1)}$$
$$y = mx + c \text{ ......(2)}$$

Take Equation (1) = Equation (2),

$$2x^2 + 5x + 8 = mx + c$$
$$2x^2 + (5 - m)x + (8 - c) = 0$$

Given that the line does not intersect the curve, $b^2 - 4ac < 0$

$$(5 - m)^2 - 4(2)(8 - c) < 0$$
$$25 - 10m + m^2 - 64 + 8c < 0$$
$$m^2 - 10m - 39 + 8c < 0 \textbf{ (shown)}$$

□

(b) Given that the solution set is $-5 < m < 15$

$$(m + 5)(m - 15) = m^2 - 10m - 75$$

Comparing coefficients,

$$-75 = -39 + 8c$$
$$c = \boldsymbol{-4\frac{1}{2}}$$

2 Surds

2.1 Full Solutions

1. (a)

$$\begin{aligned}\text{Cross-sectional area} &= \pi\left(4\sqrt{3}-1\right)^2 - \pi\left(3\sqrt{3}-1\right)^2 \\ &= \pi\left(48-8\sqrt{3}+1\right) - \pi\left(27-6\sqrt{3}+1\right) \\ &= \mathbf{\left(21-2\sqrt{3}\right)\boldsymbol{\pi}\ cm^2}\end{aligned}$$

(b)

$$\begin{aligned}\text{Volume} &= \left(521\sqrt{3}-108\right)\pi \\ \pi\left(21-2\sqrt{3}\right)\left(c+d\sqrt{3}\right) &= \left(521\sqrt{3}-108\right)\pi \\ \left(c+d\sqrt{3}\right) &= \frac{\left(521\sqrt{3}-108\right)\pi}{\left(21-2\sqrt{3}\right)\pi} \\ &= \frac{521-108\sqrt{3}}{21-2\sqrt{3}} \times \frac{21+2\sqrt{3}}{21+2\sqrt{3}} \\ &= \frac{10941\sqrt{3}+3126-2268+216\sqrt{3}}{\left(21-2\sqrt{3}\right)\left(21+2\sqrt{3}\right)} \\ &= \frac{10725\sqrt{3}+858}{429} \\ &= \mathbf{\left(25\sqrt{3}+2\right)\ cm}\end{aligned}$$

2. (a)

$$\begin{aligned}
\text{Area} &= \pi\left(\frac{3}{\sqrt{6}}+\sqrt{3}\right)^2 \\
&= \pi\left(\frac{3+3\sqrt{2}}{\sqrt{6}}\right)^2 \\
&= \pi\left(\frac{9+18\sqrt{2}+18}{6}\right) \\
&= \frac{\left(\mathbf{9+6\sqrt{2}}\right)\boldsymbol{\pi}}{\mathbf{2}}\ \mathbf{cm^2}
\end{aligned}$$

(b) Let the height be h

$$\begin{aligned}
\text{Surface Area} &= \pi\left(20\sqrt{2}+10\right) \\
2\pi\left(\frac{3}{\sqrt{6}}+\sqrt{3}\right)h &= \pi\left(20\sqrt{2}+10\right) \\
\left(\sqrt{6}+2\sqrt{3}\right)h &= 20\sqrt{2}+10 \\
h &= \frac{20\sqrt{2}+10}{\sqrt{6}+2\sqrt{3}}\times\frac{\sqrt{6}-2\sqrt{3}}{\sqrt{6}-2\sqrt{3}} \\
&= \frac{20\sqrt{12}-40\sqrt{6}+10\sqrt{6}-20\sqrt{3}}{6-12} \\
&= \frac{20\sqrt{3}-30\sqrt{6}}{-6} \\
&= \left(\mathbf{5\sqrt{6}-\frac{10}{3}\sqrt{3}}\right)\ \mathbf{cm}
\end{aligned}$$

3.

$$\begin{aligned}\sqrt{a+b\sqrt{3}} &= \frac{2\sqrt{3}}{3-\sqrt{3}} \times \frac{3+\sqrt{3}}{3+\sqrt{3}} \\ &= \frac{6\sqrt{3}+2(3)}{6} \\ &= \sqrt{3}+1\end{aligned}$$

$$\begin{aligned}\therefore a+b\sqrt{3} &= \left(\sqrt{3}+1\right)^2 \\ &= 3+2\sqrt{3}+1 \\ &= 4+2\sqrt{3}\end{aligned}$$

$$\therefore \boldsymbol{a=4} \qquad \boldsymbol{b=2}$$

4. (a)

$$\begin{aligned}CM &= \sqrt{12^2-6^2} \\ &= \sqrt{108} \\ &= 6\sqrt{3} \text{ cm}\end{aligned}$$

$$\begin{aligned}\therefore \text{Time taken} &= 6\sqrt{3} \div \frac{6-3\sqrt{3}}{4} \\ &= \frac{24\sqrt{3}}{6-3\sqrt{3}} \times \frac{6+3\sqrt{3}}{6+3\sqrt{3}} \\ &= \frac{144\sqrt{3}+216}{\left(6-3\sqrt{3}\right)\left(6+3\sqrt{3}\right)} \\ &= \frac{144\sqrt{3}+216}{9} \\ &= \left(\mathbf{16\sqrt{3}+24}\right) \textbf{ seconds}\end{aligned}$$

(b)

$$\begin{aligned}125^k &= \sqrt[3]{25\sqrt{5}} \\ 5^{3k} &= \sqrt[3]{5^{2\frac{1}{2}}} \\ &= \left(5^{2\frac{1}{2}}\right)^{\frac{1}{3}} \\ &= 5^{\frac{5}{6}}\end{aligned}$$

Comparing coefficients

$$\begin{aligned}\therefore 3k &= \frac{5}{6} \\ k &= \frac{\mathbf{5}}{\mathbf{18}}\end{aligned}$$

5. (a)

$$\begin{aligned}\text{LHS} &= \frac{15^{2k} \times 9^{4k} \times 5^{6k}}{3^{2k}} \\ &= \frac{3^{2k} \times 5^{2k} \times 3^{8k} \times 5^{6k}}{3^{2k}} \\ &= 3^{8k} \times 5^{8k} \\ &= 15^{8k}\end{aligned}$$

$$\therefore \boldsymbol{m = 15}$$

(b)

$$\begin{aligned}\text{LHS} &= \left(\frac{4}{\sqrt{3}} + \frac{2\sqrt{15}}{3} - \frac{8}{\sqrt{12}}\right) \times \sqrt{6} \\ &= \left(\frac{4\sqrt{3}}{3} + \frac{2\sqrt{3}\sqrt{5}}{3} - \frac{8}{2\sqrt{3}}\right) \times \sqrt{3}\sqrt{2} \\ &= \left(\frac{4\sqrt{3} + 2\sqrt{3}\sqrt{5}}{3} - \frac{4}{\sqrt{3}}\right) \times \sqrt{3}\sqrt{2} \\ &= \left(\frac{4\sqrt{3} + 2\sqrt{3}\sqrt{5} - 4\sqrt{3}}{3}\right) \times \sqrt{3}\sqrt{2} \\ &= \frac{2\sqrt{3}\sqrt{5}}{3} \times \sqrt{3}\sqrt{2} \\ &= 2\sqrt{10}\end{aligned}$$

$$\therefore \boldsymbol{k = 10}$$

(c) (i) Let the intersection between the two diagonals be M. By Pythagoras' Theorem,

$$PQ^2 = PM^2 + QM^2$$

$$\begin{aligned}PQ^2 &= \left(\frac{1}{2}\left(4 + 2\sqrt{3}\right)\right)^2 + \left(\frac{1}{2}\left(6 + \frac{4}{\sqrt{3}}\right)\right)^2 \\ &= \left(2 + \sqrt{3}\right)^2 + \left(3 + \frac{2}{\sqrt{3}}\right)^2 \\ &= 4 + 4\sqrt{3} + 3 + 9 + \frac{12}{\sqrt{3}} + \frac{4}{3} \\ &= \boldsymbol{\frac{52}{3} + 8\sqrt{3}}\end{aligned}$$

(ii)

$$\begin{aligned}\text{Area of } \triangle PQR &= \frac{1}{4} \times PR \times QS \\ &= \frac{1}{4}\left(4 + 2\sqrt{3}\right)\left(6 + \frac{4}{\sqrt{3}}\right) \\ &= \frac{1}{4}\left(24 + \frac{16}{\sqrt{3}} + 12\sqrt{3} + 8\right) \\ &= \frac{1}{4}\left(32 + \frac{52}{3}\sqrt{3}\right) \\ &= \boldsymbol{\left(8 + \frac{13\sqrt{3}}{3}\right)\ \text{cm}^2}\end{aligned}$$

3 Polynomials

3.1 Full Solutions

1. (a)

$$x^2 + 2x - 3 = (x+3)(x-1)$$

Let $f(1) = 0$

$$\begin{aligned}(1)^4 + 6(1)^3 + 2a(1)^2 + b(1) - 3(a) &= 0 \\ b - a &= -7 \text{ ......(1)}\end{aligned}$$

Let $f(-3) = 0$

$$\begin{aligned}(-3)^4 + 6(-3)^3 + 2a(-3)^2 + b(-3) - 3(a) &= 0 \\ 15a - 3b &= 81 \\ 5a - b &= 27 \text{ ......(2)}\end{aligned}$$

Take Equation (1) + Equation (2),

$$\begin{aligned}(b-a) + (5a-b) &= -7 + 27 \\ 4a &= 20 \\ a &= 5\end{aligned}$$

Substitute $a = 5$ into Equation (1),

$$\begin{aligned}b - 5 &= -7 \\ b &= -2\end{aligned}$$

$$\therefore \boldsymbol{a = 5} \qquad \boldsymbol{b = -2}$$

(b)

$$\begin{aligned}f(x) &= x^4 + 6x^3 + 10x^2 - 2x - 15 \\ &= \left(x^2 + 2x - 3\right)\left(x^2 + cx + 5\right)\end{aligned}$$

Comparing x^2coefficients,

$$\begin{aligned}-3 + 2c + 5 &= 10 \\ 2c &= 8 \\ c &= 4\end{aligned}$$

$$\begin{aligned}\therefore f(x) &= (x^2 + 2x - 3)(x^2 + 4x + 5) \\ &= (x+3)(x-1)(x^2 + 4x + 5)\end{aligned}$$

For $x^2 + 4x + 5$,

$$\begin{aligned}b^2 - 4ac &= (4)^2 - 4(1)(5) \\ &= -4 < 0\end{aligned}$$

Since the discriminant < 0, $x^2 + 4x + 5$ has no real roots

$\therefore$ Number of real roots is **2**

2. Let $x = -2$,

$$(-2)^3 - 4(-2)^2 - 8(-2) + 8 = 0$$

$$\therefore (x+2) \text{ is a factor of } f(x)$$

$$f(x) = (x+2)(x^2 + cx + 4)$$

Comparing x^2 coefficients,

$$\begin{aligned} 2 + c &= -4 \\ c &= -6 \end{aligned}$$

$$\therefore f(x) = (x+2)(x^2 - 6x + 4) = 0$$

For $x^2 - 6x + 4$,

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)} \\ &= \frac{6 \pm \sqrt{20}}{2} \\ &= \frac{6 \pm 2\sqrt{5}}{2} \\ &= 3 \pm \sqrt{5} \end{aligned}$$

$$\therefore \boldsymbol{x = 2} \qquad \boldsymbol{x = 3 \pm \sqrt{5}}$$

3. (a) Let $f(1) = 0$,

$$\begin{aligned} 3(1)^3 + a(1)^2 + b(1) + 2 &= 0 \\ a + b &= -5 \\ a &= -5 - b \text{ ......(1)} \end{aligned}$$

$$\begin{aligned} f(2) &= \left(2\frac{1}{2}\right) f(-1) \\ 3(2)^3 + a(2)^2 + b(2) + 2 &= \left(2\frac{1}{2}\right) \left[3(-1)^3 + a(-1)^2 + b(-1) + 2\right] \\ 26 + 4a + 2b &= \left(2\frac{1}{2}\right) [-1 + a - b] \\ 1\frac{1}{2}a + 4\frac{1}{2}b &= -28\frac{1}{2} \text{ ......(2)} \end{aligned}$$

Substitute Equation (1) into Equation (2),

$$\begin{aligned} 1\frac{1}{2}(-5 - b) + 4\frac{1}{2}b &= -28\frac{1}{2} \\ 3b &= -21 \\ b &= -7 \end{aligned}$$

Substitute $b = -7$ into Equation (1),

$$\begin{aligned} a &= -5 - (-7) \\ &= 2 \end{aligned}$$

$$\therefore a = 2 \qquad b = -7 \textbf{ (shown)}$$

(b) Let c be a constant

$$\begin{aligned} f(x) &= 3x^3 + 2x^2 - 7x + 2 \\ &= (x - 1)(3x^2 + cx - 2) \end{aligned}$$

Compare coefficients,

$$\begin{aligned} 2 &= 3(-1) + c \\ c &= 5 \end{aligned}$$

Since $f(x) = 0$

$$\begin{aligned} (x - 1)(3x^2 + 5x - 2) &= 0 \\ (x - 1)(3x - 1)(x + 2) &= 0 \end{aligned}$$

$$\therefore \boldsymbol{x = 1 \quad \text{or} \quad x = \frac{1}{3} \quad \text{or} \quad x = -2}$$

(c)

$$\begin{aligned}
3\sin^2 y - 2\sec y - 2\cos y + 4 &= 0 \\
3\left(1-\cos^2 y\right) - \frac{2}{\cos y} - 2\cos y + 4 &= 0 \\
3\cos y - 3\cos^3 y - 2 - 2\cos^2 y + 4\cos y &= 0 \\
3\cos^3 y + 2\cos^2 y - 7\cos y + 2 &= 0
\end{aligned}$$

Comparing the 2 equations,

$$x = \cos y$$

$$\cos y = 1 \qquad \text{or} \qquad \cos y = \frac{1}{3} \qquad \text{or} \qquad \cos y = -2 \text{ (N.A.)}$$

For $\cos y = 1$

$$\alpha = 0 \quad \text{(Quadrant 1 or 4)}$$

$$y = \mathbf{0^\circ} \quad \text{or} \quad y = \mathbf{360^\circ}$$

For $\cos y = \dfrac{1}{3}$

$$\alpha = \cos^{-1}\left(\frac{1}{3}\right) \quad \text{(Quadrant 1 or 4)}$$

$$\begin{aligned}
y &= \cos^{-1}\left(\frac{1}{3}\right) \\
&= \mathbf{70.5^\circ\ (1.d.p.)}
\end{aligned}$$

$$\begin{aligned}
y &= 360^\circ - \cos^{-1}\left(\frac{1}{3}\right) \\
&= \mathbf{289.5^\circ\ (1.d.p.)}
\end{aligned}$$

$$\boldsymbol{y = 0^\circ \qquad y = 70.5^\circ \qquad y = 289.5^\circ \qquad y = 360^\circ}$$

4.

$$\begin{aligned}
f(x) &= 2\left(7^{n+2}\right) + 7^n + 3\left(7^{n+1}\right) \\
&= 2(49)\left(7^n\right) + 7^n + 21\left(7^n\right) \\
&= 120\left(7^n\right) \\
&= 10(12)\left(7^n\right)
\end{aligned}$$

Since 10 is a factor of $f(x)$, Billy's comment is **correct**

□

5. (a) By long division

$$Q(x) = \mathbf{2x^2 - x - 3}$$

(b)

$$\begin{aligned} f(x) &= 2x^4 + 5x^3 - 8x^2 - 8x + 3 \\ &= \left(x^2 + 3x - 1\right)\left(2x^2 - x - 3\right) \\ &= \mathbf{\left(x^2 + 3x - 1\right)(2x - 3)(x + 1)} \end{aligned}$$

(c) By observation

$$\begin{aligned} 32p^4 + 40p^3 - 32p^2 - 16p + 3 &= 0 \\ 2(2p)^4 + 5(2p)^3 - 8(2p)^2 - 8(2p) + 3 &= 0 \end{aligned}$$

$$x = 2p$$

$$\begin{aligned} (2p)^2 + 3(2p) - 1 &= 0 \qquad \text{or} \qquad (2(2p) - 3)((2p) + 1) = 0 \\ 4p^2 + 6p - 1 &= 0 \qquad \text{or} \qquad (4p - 3)(2p + 1) = 0 \end{aligned}$$

For the quadratic factor

$$\begin{aligned} p &= \frac{-6 \pm \sqrt{(6)^2 - 4(4)(-1)}}{2(4)} \\ &= \mathbf{\frac{-3 \pm \sqrt{13}}{4}} \end{aligned}$$

For the linear factors

$$p = \mathbf{\frac{3}{4}} \qquad \text{or} \qquad p = \mathbf{-\frac{1}{2}}$$

4 Partial Fractions

4.1 Full Solutions

1. (a)

$$\frac{P(x)}{Q(x)} = \frac{3x^3 - 9x^2 - 18x + 24}{x^2 - 9}$$

By long division,

$$\frac{P(x)}{Q(x)} = 3x - 9 + \frac{9x - 57}{x^2 - 9}$$

Hence,

$$\frac{9x - 57}{x^2 - 9} = \frac{A}{x - 3} + \frac{B}{x + 3}$$
$$9x - 57 = A(x + 3) + B(x - 3)$$

Let $x = 3$,

$$9(3) - 57 = 6A$$
$$A = 5$$

Let $x = -3$,

$$9(-3) - 57 = -6B$$
$$B = 14$$

$$\frac{P(x)}{Q(x)} = \mathbf{3x - 9 + \frac{5}{x - 3} + \frac{14}{x + 3}}$$

(b) (i)

$$3x^4 - 9x^2 - 18x + 24 = 0$$
$$x^3 - 3x^2 - 6x + 8 = 0$$
$$(x + 2)(x - 4)(x - 1) = 0$$

$$\therefore \boldsymbol{x = -2} \quad \text{or} \quad \boldsymbol{x = 4} \quad \text{or} \quad \boldsymbol{x = 1}$$

(ii) By comparing the equations

$$x = \log_2 \sqrt{y}$$

$$\log_2 \sqrt{y} = -2 \qquad \log_2 \sqrt{y} = 4 \qquad \log_2 \sqrt{y} = 1$$
$$\sqrt{y} = 2^{-2} \qquad \sqrt{y} = 2^4 \qquad \sqrt{y} = 2$$
$$y = \mathbf{2^{-4}} \qquad y = \mathbf{2^8} \qquad y = \mathbf{2^2}$$

2.

$$\frac{4}{\left(x^2+4\right)(x-2)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+4}$$
$$4=A\left(x^2+4\right)+(Bx+C)(x-2)$$

Let $x=2$,

$$4=8A$$
$$A=\frac{1}{2}$$

Let $x=0$,

$$4=4\left(\frac{1}{2}\right)-2C$$
$$C=-1$$

Let $x=1$,

$$4=5\left(\frac{1}{2}\right)-(B-1)$$
$$B=-\frac{1}{2}$$

$$\therefore \frac{4}{\left(x^2+4\right)(x-2)}=\boldsymbol{\frac{1}{2(x-2)}-\frac{x+2}{2\left(x^2+4\right)}}$$

3. By Long Division,

$$\frac{2x^3-3x-1}{(x+3)(x-1)}=2x-4+\frac{11x-13}{(x+3)(x-1)}$$

Hence,

$$\frac{11x-13}{(x+3)(x-1)}=\frac{A}{x+3}+\frac{B}{x-1}$$
$$11x-13=A(x-1)+B(x+3)$$

Let $x=1$,

$$11(1)-13=4B$$
$$B=-\frac{1}{2}$$

Let $x=-3$,

$$11(-3)-13=-4A$$
$$A=\frac{23}{2}$$

$$\therefore \frac{2x^3-3x-1}{(x+3)(x-1)}=\boldsymbol{2x-4+\frac{23}{2(x+3)}-\frac{1}{2(x-1)}}$$

4.

$$\frac{8x^2-2x+19}{(1-x)\left(4+x^2\right)}=\frac{A}{1-x}+\frac{Bx+C}{4+x^2}$$
$$8x^2-2x+19=A\left(4+x^2\right)+(Bx+C)(1-x)$$

Let $x=1$,

$$8(1)^2-2(1)+19=5A$$
$$A=5$$

Let $x=0$,

$$8(0)^2-2(0)+19=4(5)+C$$
$$C=-1$$

Comparing coefficient of x^2 terms,

$$8=5-B$$
$$B=-3$$

$$\therefore \frac{8x^2-2x+19}{(1-x)\left(4+x^2\right)}=\mathbf{\frac{5}{1-x}-\frac{3x+1}{4+x^2}}$$

5. (a) By factor theorem and long division

$$f(x)=\mathbf{(x-3)^2(2x+1)}$$

(b) By Long Division,

$$\frac{6x^3-33x^2+35x+51}{2x^3-11x^2+12x+9}=3+\frac{24-x}{(x-3)^2(2x+1)}$$

Hence,

$$\frac{24-x}{(x-3)^2(2x+1)}=\frac{A}{2x+1}+\frac{B}{x-3}+\frac{C}{(x-3)^2}$$
$$-x+24=A(x-3)^2+B(x-3)(2x+1)+C(2x+1)$$

Let $x=3$,

$$-(3)+24=7C$$
$$C=3$$

Let $x=-\frac{1}{2}$

$$-\left(-\frac{1}{2}\right)+24=\frac{49}{4}A$$
$$A=2$$

Let $x=0$,

$$24=9(2)-3B+3$$
$$B=-1$$

$$\therefore \frac{6x^3-33x^2+35x+51}{2x^3-11x^2+12x+9}=\mathbf{3+\frac{2}{2x+1}-\frac{1}{x-3}+\frac{3}{(x-3)^2}}$$

5 Binomial Theorem

5.1 Full Solutions

1. (a)

$$\begin{aligned} T_{r+1} &= \binom{10}{r}\left(x^2\right)^{10-x}\left(-\frac{1}{2x^3}\right)^r \\ &= \binom{10}{r}\left(-\frac{1}{2}\right)^r\left(x^{20-5r}\right) \end{aligned}$$

For the independent term of x, x^0

$$\begin{aligned} 20-5r &= 0 \\ r &= 4 \end{aligned}$$

Hence,

$$\begin{aligned} \text{Independent term of } x &= \binom{10}{4}\left(-\frac{1}{2}\right)^4 \\ &= \mathbf{13\frac{1}{8}} \end{aligned}$$

(b) (i) (a)

$$(2-3x)^7 = \mathbf{128 - 1344x + 6048x^2 + ...}$$

(b)

$$\left(1+\frac{x}{3}\right)^7 = \mathbf{1 + \frac{7}{3}x + \frac{7}{3}x^2 + ...}$$

(ii)

$$\begin{aligned} \left(2-\frac{7}{3}x-x^2\right)^7 &= \left[(2-3x)\left(1+\frac{x}{3}\right)\right]^7 \\ &= \left(128-1344x+6048x^2+...\right)\left(1+\frac{7}{3}x+\frac{7}{3}x^2+...\right) \\ &= ... + \left[128\left(\frac{7}{3}\right) - 1134\left(\frac{7}{3}\right) + 6048\right]x^2 + ... \\ &= ... + 3210\frac{2}{3}x^2 + ... \end{aligned}$$

$$\text{Coefficient of } x^2 = \mathbf{3210\frac{2}{3}}$$

2. (a)

$$\begin{aligned}\text{LHS} &= \left(1+ax+bx^2\right)^8 \\ &= 1^8 + \binom{8}{1}\left(1^7\right)\left(ax+bx^2\right) + \binom{8}{2}\left(1^6\right)\left(ax+bx^2\right)^2 + ... \\ &= 1 + 8\left(ax+bx^2\right) + 28\left(a^2x^2 + ...\right) + ... \\ &= 1 + 8ax + 8bx^2 + 28a^2x^2 + ...\end{aligned}$$

Comparing terms

$$\begin{aligned}-40 &= 8a \\ a &= -5\end{aligned}$$

$$\begin{aligned}8b + 28(-5)^2 &= 748 \\ b &= 6\end{aligned}$$

(b)

$$\begin{aligned}T_{r+1} &= \binom{16}{r}\left(x^2\right)^{16-r}\left(-\frac{1}{2x^6}\right)^r \\ &= \binom{16}{r}\left(-\frac{1}{2}\right)^r\left(x^{32-8r}\right)\end{aligned}$$

For the independent term of x, x^0

$$\begin{aligned}32 - 8r &= 0 \\ r &= 4\end{aligned}$$

Hence,

$$\begin{aligned}\text{Independent term of } x &= \binom{16}{4}\left(-\frac{1}{2}\right)^4 \\ &= \mathbf{113\frac{3}{4}}\end{aligned}$$

(c) (i)

$$T_{r+1} = \binom{9}{r}(x)^{9-r}\left(\frac{k}{x}\right)^r$$
$$= \binom{9}{r}(k)^r\left(x^{9-2r}\right)$$

For x term,

$$9-2r = 1$$
$$r = 4$$

For x^3 term,

$$9-2r = 3$$
$$r = 3$$

Since the coefficients are the same,

$$\binom{9}{4}(k)^4 = \binom{9}{3}(k)^3$$
$$k = \mathbf{\frac{2}{3}}$$

(ii)

$$\left(1-3x^2\right)\left(x+\frac{k}{x}\right)^9 = \left(1-3x^2\right)\left[... + \binom{9}{3}(x)^6\left(\frac{2}{3x}\right)^3 + \binom{9}{4}(x)^5\left(\frac{2}{3x}\right)^4 + ...\right]$$
$$= \left(1-3x^2\right)\left[... + \frac{224}{9}x^3 + \frac{224}{9}x + ...\right]$$

Hence,

$$\text{Coefficient of } x^3 = (1)\left(\frac{224}{9}\right) + (-3)\left(\frac{224}{9}\right)$$
$$= \mathbf{-49\frac{7}{9}}$$

3. (a) (i)

$$(1+a)^8 = \mathbf{1 + 8a + 28a^2 + 56a^3 + ...}$$

(ii)

$$\begin{aligned}\left(1+x+x^2\right)^8 &= 1 + 8\left(x+x^2\right) + 28\left(x+x^2\right)^2 + 56\left(x+x^2\right)^3 + ... \\ &= 1 + 8x + 8x^2 + 28\left(x^2 + 2x^3 + ...\right) + 56\left(x^3 + ...\right) \\ &= \mathbf{1 + 8x + 36x^2 + 112x^3 + ...}\end{aligned}$$

(iii) By comparing

$$\left(1+x+x^2\right)^8 \qquad 1.0101^8$$

we can see that

$$x = 0.01$$

Hence,

$$\begin{aligned}1.0101^8 &= 1 + 8(0.01) + 36(0.01)^2 + 112(0.01)^3 + ... \\ &= \mathbf{1.083712\ (6.d.p.)}\end{aligned}$$

(b) (i)

$$T_{r+1} = \boldsymbol{\binom{12}{r}(3x)^{12-r}\left(-\frac{2}{x^2}\right)^r}$$

(ii)

$$\begin{aligned}T_{r+1} &= \boldsymbol{\binom{12}{r}(3x)^{12-r}\left(-\frac{2}{x^2}\right)^r} \\ &= \binom{12}{r}(3)^{12-r}(-2)^r(x)^{12-3r}\end{aligned}$$

$$\therefore \text{Power of } x = \mathbf{12 - 3r}$$

(iii) For the x^5 term,

$$\begin{aligned}12 - 3r &= 5 \\ r &= \frac{7}{3} \notin \mathbb{Z}^+ \quad \Rightarrow\Leftarrow\end{aligned}$$

Since r is not an integer, there is no x^5 term

□

4. (a)

$$\begin{aligned}(3x-1)(1-kx)^7 &= (3x-1)\left[(1)^7 + \binom{7}{1}(1)^6(-kx) + \binom{7}{2}(1)^5(-kx)^2 + ...\right] \\ &= (3x-1)\left(1 - 7kx + 21k^2x^2 + ...\right)\end{aligned}$$

Since there is no x^2 term,

$$\begin{aligned}-7k(3) + \left(21k^2\right)(-1) &= 0 \\ -21k(1+k) &= 0\end{aligned}$$

$$\therefore k = 0 \text{ (N.A.)} \qquad \text{or} \qquad k = -1$$

(b)

$$\begin{aligned}T_{r+1} &= \binom{12}{r}\left(\frac{2}{x^3}\right)^{12-r}\left(-x^2\right)^r \\ &= \binom{12}{r}\left(2^{12-r}\right)(-1)^r x^{5r-36}\end{aligned}$$

Since we are looking for the power of x first becomes positive,

$$\begin{aligned}5r - 36 &> 0 \\ r &> 7.2 \\ &\approx 8\end{aligned}$$

$$\begin{aligned}\therefore T_9 &= \binom{12}{8}\left(2^4\right)(-1)^8 x^{40-36} \\ &= \mathbf{7920x^4}\end{aligned}$$

5. (a)

$$T_{r+1} = \binom{8}{r}(3)^{8-r}\left(-2x^2\right)^r$$
$$= \binom{8}{r}(3)^{8-r}(-2)^r x^{2r}$$

For the x^{10} term,

$$2r = 10$$
$$r = 5$$

$$\text{Coefficient} = \binom{8}{5}(3)^{8-5}(-2)^5$$
$$= \mathbf{-48384}$$

(b)

$$(1+3x)^m = 1 + \binom{m}{1}(1)^{m-1}(3x) + \binom{m}{2}(1)^{m-2}(3x)^2 + ...$$
$$= 1 + 3mx + \frac{9m(m-1)}{2}x^2 + ...$$

Since the difference is 462,

$$\frac{9m(m-1)}{2} - 3m = 462$$
$$9m^2 - 15m - 924 = 0$$
$$3m^2 - 5m - 308 = 0$$
$$(3m+28)(m-11) = 0$$

$$\therefore m = -\frac{28}{3} \text{ (rej.)} \qquad \text{or} \qquad m = \mathbf{11}$$

6 Exponential & Logarithms

6.1 Full Solutions

1. (a) When $t = 0$,

$$\begin{aligned} P &= 300\left(2 + 5e^{-k(0)}\right) \\ &= 300(2+5) \\ &= \mathbf{2100} \end{aligned}$$

(b) When $t = 3$, $P = 2400$

$$\begin{aligned} 2400 &= 300\left(2 + 5e^{-3k}\right) \\ 6 &= 5e^{-3k} \\ e^{-3k} &= \frac{6}{5} \\ k &= -\frac{1}{3}\ln\left(\frac{6}{5}\right) \\ &= -0.0607738... \\ &= \mathbf{-0.0608\ (3.s.f.)} \end{aligned}$$

(c) When $t = 5$,

$$\begin{aligned} P &= 300\left(2 + 5e^{\frac{5}{3}\ln\left(\frac{6}{5}\right)}\right) \\ &= 2632.637... > 1000 \end{aligned}$$

Not necessary to replenish

□

2. (a) When $P_0 = 20000$, $P_n = 22497.28$, $t = 3$

$$\begin{aligned} 22497.28 &= 20000\left(1 + \frac{r}{100}\right)^3 \\ \left(1 + \frac{r}{100}\right)^3 &= 1.124864 \\ 1 + \frac{r}{100} &= 1.04 \\ r &= \mathbf{4} \end{aligned}$$

(b) Since Mandy wants to double the principal amount,

$$\begin{aligned} \left(1 + \frac{4}{100}\right)^n &= 2 \\ 1.04^n &= 2 \\ n &= \frac{\lg 2}{\lg 1.04} \\ &= 17.672987... \\ &= \mathbf{17.7\ years\ (3.s.f.)} \end{aligned}$$

3. (a)

$$\begin{aligned}\log_3 2 \times \log_4 3 \times \log_5 4 \times ... \times \log_{n+1} n &= \frac{\lg 2}{\lg 3} \times \frac{\lg 3}{\lg 4} \times \frac{\lg 4}{\lg 5} \times ... \times \frac{\lg n}{\lg(n+1)} \\ &= \boldsymbol{\frac{\lg 2}{\lg(n+1)}}\end{aligned}$$

(b)

$$\begin{aligned}6^{x+1} - 6^{1-x} &= 5 \\ 6\left(6^x\right) - \frac{6}{6^x} &= 5\end{aligned}$$

Let $u = 6^x$

$$\begin{aligned}6u - \frac{6}{u} - 5 &= 0 \\ 6u^2 - 5u - 6 &= 0 \\ (3u+2)(2u-3) &= 0\end{aligned}$$

$$\therefore u = \frac{3}{2} \qquad \text{or} \qquad u = -\frac{2}{3} \text{ (rej)}$$

Hence,

$$\begin{aligned}6^x &= \frac{3}{2} \\ x &= \frac{\lg\left(\frac{3}{2}\right)}{\lg 6} \\ &= 0.226294... \\ &= \mathbf{0.226\ (3.s.f.)}\end{aligned}$$

4. (a)

$$\begin{aligned}
2\log_2(1-x) - \log_2 x - 2 &= \log_2 2x + 1 \\
\log_2\left[\frac{(1-x)^2}{x}\right] - \log_2 2x &= 3 \\
\log_2\left[\frac{\frac{(1-x)^2}{x}}{2x}\right] &= 3 \\
\frac{(1-x)^2}{2x^2} &= 2^3 \\
1 - 2x + x^2 &= 16x^2 \\
15x^2 + 2x - 1 &= 0 \\
(5x-1)(3x+1) &= 0
\end{aligned}$$

$$\therefore x = \mathbf{\frac{1}{5}} \quad \text{or} \quad x = -\frac{1}{3} \text{ (rej.)}$$

(b)

$$\begin{aligned}
\frac{(\log_x y)^3}{\log_y x} - 20 &= 61 \\
\frac{(\log_x y)^3}{\left(\frac{1}{\log_x y}\right)} &= 81 \\
(\log_x y)^4 &= 81
\end{aligned}$$

$$\log_x y = 3 \quad \text{or} \quad \log_x y = -3$$

$$\boldsymbol{y = x^3} \quad \text{or} \quad \boldsymbol{y = \frac{1}{x^3}}$$

5. (a)

$$3\log_3 x - \log_x 3 = 2$$
$$3\log_3 x - \frac{1}{\log_3 x} = 2$$

Let $u = \log_3 x$,

$$3u - \frac{1}{u} = 2$$
$$3u^2 - 2u - 1 = 0$$
$$(u-1)(3u+1) = 0$$

$$u = 1 \quad \text{or} \quad u = -\frac{1}{3}$$
$$\log_3 x = 1 \quad \text{or} \quad \log_3 x = -\frac{1}{3}$$
$$x = \mathbf{3} \quad \text{or} \quad x = \mathbf{3^{-\frac{1}{3}}}$$

(b)

$$2\log_2(1-2x) - \log_2(6-5x) = 0$$
$$\log_2(1-2x)^2 = \log_2(6-5x)$$
$$(1-2x)^2 = 6-5x$$
$$1 - 4x + 4x^2 - 6 + 5x = 0$$
$$4x^2 + x - 5 = 0$$
$$(x-1)(4x+5) = 0$$

$$\therefore x = 1 \text{ (rej)} \quad \text{or} \quad x = \mathbf{-\frac{5}{4}}$$

7 Trigonometry

7.1 Full Solutions

1. (a) (i)

$$\begin{aligned}\text{LHS} &= \sin(A+B)\sin(A-B)\\ &= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)\\ &= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B\\ &= \sin^2 A\left(1-\sin^2 B\right) - \sin^2 B\left(1-\sin^2 A\right)\\ &= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B\\ &= \sin^2 A - \sin^2 B\\ &= \text{RHS } \textbf{(shown)}\end{aligned}$$

□

(ii)

$$\begin{aligned}\sin\left(\frac{7\pi}{12}\right)\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\ &= \sin^2\left(\frac{\pi}{3}\right) - \sin^2\left(\frac{\pi}{4}\right)\\ &= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2\\ &= \mathbf{\frac{1}{4}}\end{aligned}$$

(b) (i)

$$\begin{aligned}\text{LHS} &= \frac{\sec^2 x + 2\tan x}{1+2\sin x\cos x}\\ &= \frac{\left(\frac{1}{\cos^2 x}+\frac{2\sin x}{\cos x}\right)}{1+2\sin x\cos x}\\ &= \frac{\left(\frac{1+2\sin x\cos x}{\cos^2 x}\right)}{1+2\sin x\cos x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x\\ &= \text{RHS } \textbf{(shown)}\end{aligned}$$

□

(ii) Comparing part (b)(i) and (b)(ii),

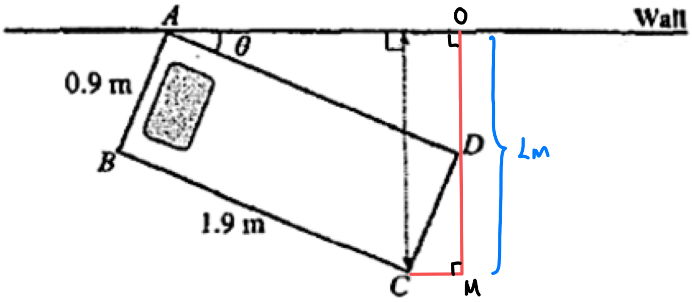
$$\sec^2\left(x-\frac{\pi}{3}\right) = \frac{4}{3}$$
$$\cos\left(x-\frac{\pi}{3}\right) = \pm\frac{\sqrt{3}}{2}$$

By solving this,

$$\alpha = \frac{\pi}{6}$$

$$\therefore \boldsymbol{x = \frac{\pi}{6} \qquad x = \frac{\pi}{2} \qquad x = \frac{7\pi}{6} \qquad x = \frac{3\pi}{2}}$$

2. (a) Draw a line as shown and let the new points be O and M



In $\triangle ODA$,

$$\sin\theta = \frac{OD}{AD}$$
$$OD = 1.9\sin\theta$$

In $\triangle CDM$,

$$\cos\theta = \frac{DM}{DC}$$
$$DM = 0.9\cos\theta$$

$$\therefore L = OD + DM$$
$$= 1.9\sin\theta + 0.9\cos\theta \ \textbf{(shown)}$$

□

(b)

$$R = \sqrt{(1.9)^2 + (0.9)^2}$$
$$= \sqrt{4.42}$$

$$\alpha = \tan^{-1}\left(\frac{0.9}{1.9}\right)$$
$$= 25.346175...$$
$$= 25.3^\circ \ (1.\text{d.p.})$$

$$\therefore L = \boldsymbol{\sqrt{4.42}\sin(\theta + 25.3^\circ)}$$

(c) At maximum L,

$$\begin{aligned} L &= \sqrt{4.42} \\ &= 2.102379... \\ &= \mathbf{2.10\ m\ (3.s.f.)} \end{aligned}$$

This occurs when

$$\begin{aligned} \sin(\theta + 25.346°) &= 1 \\ \therefore \theta &= 90° - \tan^{-1}\left(\frac{0.9}{1.9}\right) \\ &= 64.653824... \\ &= \mathbf{64.7°\ (1.d.p.)} \end{aligned}$$

(d) When $L = 1.3$ m,

$$\begin{aligned} 1.3 &= \sqrt{4.42}\sin\left[\theta + \tan^{-1}\left(\frac{0.9}{1.9}\right)\right] \\ \theta + \tan^{-1}\left(\frac{0.9}{1.9}\right) &= \sin^{-1}\left(\frac{1.3}{\sqrt{4.42}}\right) \text{ (Quadrant 1)} \\ \therefore \theta &= \sin^{-1}\left(\frac{1.3}{\sqrt{4.42}}\right) - \tan^{-1}\left(\frac{0.9}{1.9}\right) \\ &= 12.849339... \\ &= \mathbf{12.8°\ (1.d.p.)} \end{aligned}$$

3. (a)

$$a = \mathbf{-4} \qquad b = \mathbf{10} \qquad c = \mathbf{3}$$

(b) When it first emerge from the water, $h = 0$,

$$\begin{aligned} -4\sin\left(\frac{\pi}{10}t\right) + 3 &= 0 \\ \sin\left(\frac{\pi}{10}t\right) &= \frac{3}{4} \end{aligned}$$

Since we are looking for the point where it first emerges from the water, 2nd quadrant

$$\begin{aligned} t &= \frac{10\left(\pi - \sin^{-1}\left(\frac{3}{4}\right)\right)}{\pi} \\ &= 7.300534... \\ &= \mathbf{7.30\ s\ (3.s.f.)} \end{aligned}$$

4. (a)

$$\angle BOC = \frac{360°}{2(12)}$$
$$= 15°$$

$$\sin \angle BOC = \frac{BC}{BO}$$
$$BC = \sin 15°$$

$$\therefore AB = 2\sin 15° \textbf{ (shown)}$$

□

(b) (i)

$$\cos 30° = \mathbf{1 - 2\sin^2 15°}$$

(ii)

$$2\sin^2 15° = 1 - \frac{\sqrt{3}}{2}$$
$$\sin^2 15° = \frac{2-\sqrt{3}}{4}$$
$$\sin 15° = \frac{1}{2}\sqrt{2-\sqrt{3}} \textbf{ (shown)}$$

□

5. (a)

$$\text{Principal values} = \boldsymbol{-\frac{\pi}{2} < x < \frac{\pi}{2}}$$

(b) Given the ratio,

$\sin A$	$\cos A$	$\tan A$
$-\dfrac{p}{\sqrt{p^2+1}}$	$\dfrac{1}{\sqrt{p^2+1}}$	$-p$

(i)

$$\sin A = \boldsymbol{-\frac{p}{\sqrt{p^2+1}}}$$

(ii)

$$\begin{aligned}\sec A &= \frac{1}{\cos A} \\ &= \boldsymbol{\frac{1}{\left(\frac{1}{\sqrt{p^2+1}}\right)}} \\ &= \boldsymbol{\sqrt{p^2+1}}\end{aligned}$$

(iii)

$$\begin{aligned}\cot(-A)^\circ &= \frac{1}{\tan(-A)} \\ &= -\frac{1}{\tan A} \\ &= -\frac{1}{(-p)} \\ &= \boldsymbol{\frac{1}{p}}\end{aligned}$$

(iv)

$$\begin{aligned}\tan(90-A)^\circ &= \cot A \\ &= \frac{1}{\tan A} \\ &= \frac{1}{(-p)} \\ &= \boldsymbol{-\frac{1}{p}}\end{aligned}$$

(c) When $x = -\frac{\pi}{12}$, $y = -4$

$$\begin{aligned}-4 &= m + 3\tan\left[3\left(-\frac{\pi}{12}\right)\right] \\ m &= \boldsymbol{-1}\end{aligned}$$

From the graph, to find n, we are in quadrant 4. Hence, at $(n, 2)$,

$$2 = -1 + 3\tan 3n$$

$$3n = \frac{\pi}{4} \quad \text{or} \quad 3n = \frac{5\pi}{4}$$

$$n = \boldsymbol{\frac{5\pi}{12}}$$

6. (a) (i)

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\frac{56}{65} = \sin A \cos B + \frac{4}{13}$$
$$\therefore \sin A \cos B = \mathbf{\frac{36}{65}}$$

(ii)

$$\frac{\tan A}{\tan B} = \frac{\left(\frac{\sin A}{\cos A}\right)}{\left(\frac{\sin B}{\cos B}\right)}$$
$$= \frac{\sin A \cos B}{\sin B \cos A}$$
$$= \frac{\left(\frac{36}{65}\right)}{\left(\frac{4}{13}\right)}$$
$$= \mathbf{\frac{9}{5}}$$

(iii) Given the ratio

$\sin(A+B)$	$\cos(A+B)$	$\tan(A+B)$
$\frac{56}{65}$	$-\frac{33}{65}$	$-\frac{33}{56}$

$$\therefore \cos(A+B) = \mathbf{-\frac{33}{65}}$$

(b) (i)

$$\begin{aligned} R &= \sqrt{(3)^2 + (1)^2} \\ &= \sqrt{10} \end{aligned}$$

$$\begin{aligned} \alpha &= \tan^{-1}\left(\frac{1}{3}\right) \\ &= 0.32175... \\ &= 0.321 \text{ (3.s.f.)} \end{aligned}$$

$$\therefore 3\sin\theta + \cos\theta = \sqrt{10}\sin(\theta + 0.322)$$

(ii) Hence, to solve the equation,

$$\begin{aligned} \sqrt{10}\sin\left[2y + \tan^{-1}\left(\frac{1}{3}\right)\right] &= 2 \\ \sin\left[2y + \tan^{-1}\left(\frac{1}{3}\right)\right] &= \frac{2}{\sqrt{10}} \end{aligned}$$

$$\alpha = \sin^{-1}\left(\frac{2}{\sqrt{10}}\right) \text{ (Quadrant 1 \& 2)}$$

For Quadrant 1,

$$\begin{aligned} y &= \frac{\sin^{-1}\left(\frac{2}{\sqrt{10}}\right) - \tan^{-1}\left(\frac{1}{3}\right)}{2} \\ &= 0.181484... \\ &= \mathbf{0.181 \text{ rad (3.s.f.)}} \end{aligned}$$

For Quadrant 2,

$$\begin{aligned} y &= \frac{\pi - \sin^{-1}\left(\frac{2}{\sqrt{10}}\right) - \tan^{-1}\left(\frac{1}{3}\right)}{2} \\ &= 1.067561... \\ &= \mathbf{1.07 \text{ rad (3.s.f.)}} \end{aligned}$$

(iii)

$$\begin{aligned} \text{Greatest value} &= \frac{1}{-\sqrt{10} + 5} \\ &= \mathbf{\frac{5 + \sqrt{10}}{15}} \end{aligned}$$

8 Coordinate Geometry

8.1 Full Solutions

1. (a) Since S is on a point on the y-axis let S be $(0, y)$, using the length of PS,

$$\sqrt{(-2-0)^2+(1-y)^2} = 2\sqrt{10}$$
$$4+1-2y+y^2 = 40$$
$$y^2-2y-35 = 0$$
$$(y-7)(y+5) = 0$$

$$y = 7 \text{ (rej)} \qquad \text{or} \qquad y = -5$$
$$\therefore S\mathbf{(0, -5)}$$

(b)

$$\text{Gradient of } PS = \frac{1-(-5)}{-2-0}$$
$$= -3$$

Since PS is perpendicular to PQ,

$$\text{Gradient of } PQ = \frac{-1}{(-3)}$$
$$= \frac{1}{3}$$

Hence, the equation of PQ is,

$$y-1 = \frac{1}{3}(x+2)$$
$$3y = x+5$$

Hence, substituting $Q(2q+1, q)$,

$$3q = (2q+1)+5$$
$$q = \mathbf{6}$$

(c) Using $q = 6$,

$$\text{Length of } PQ = \sqrt{(13+2)^2+(6-1)^2}$$
$$= \sqrt{250} \text{ units}$$

Hence, to find the area of rectangle $PQRS$,

$$\text{Area } = \sqrt{250} \times 2\sqrt{10}$$
$$= \mathbf{100 \text{ units}^2}$$

2. (a)

$$\begin{aligned} \text{Gradient of } AC &= \frac{4-(-1)}{-1-4} \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{Midpoint of } AC &= \left(\frac{-1+4}{2}, \frac{4-1}{2}\right) \\ &= \left(\frac{3}{2}, \frac{3}{2}\right) \end{aligned}$$

Note that BD and AC share the same mid-point due to the properties of a parallelogram. Also note that AC is perpendicular to BD

$$\begin{aligned} \text{Gradient of } BD &= \frac{\left(6-\frac{3}{2}\right)}{\left(p-\frac{3}{2}\right)} \\ &= -1 \\ \frac{3}{2} - p &= 6 - \frac{3}{2} \\ p &= 6 \ \textbf{(shown)} \end{aligned}$$

□

(b) Since we note that the midpoint of BD is $\left(\frac{3}{2}, \frac{3}{2}\right)$, they have the same x and y coordinate, just like point B. Hence, we can make an inference that D will have the same properties. Also note that the equation of line BD is

$$y = x$$

Let $D(a, a)$. Comparing coordinates,

$$\begin{aligned} \frac{a+6}{2} &= \frac{3}{2} \\ a &= -3 \end{aligned}$$

Hence,

$$\therefore D(\mathbf{-3, -3})$$

(c)

$$\begin{aligned} \text{Area of parallelogram } ABCD &= \frac{1}{2}\begin{vmatrix} -3 & 4 & 6 & -1 & -3 \\ -3 & -1 & 6 & 4 & -3 \end{vmatrix} \\ &= \frac{1}{2}\left|54+36\right| \\ &= \mathbf{45 \ units^2} \end{aligned}$$

3. (a) (i)

$$\text{Midpoint of } AD = \left(\frac{7-3}{2}, \frac{4+8}{2}\right)$$
$$= (2, 6)$$

$$\text{Gradient of } AD = \frac{8-4}{-3-7}$$
$$= -\frac{2}{5}$$

$$\text{Gradient of perpendicular bisector} = \frac{-1}{\left(-\frac{2}{5}\right)}$$
$$= \frac{5}{2}$$

Hence, the equation of the perpendicular bisector of AD is

$$y - 6 = \frac{5}{2}(x-2)$$
$$\boldsymbol{y = \frac{5}{2}x + 1}$$

To check if F lies on the perpendicular bisector, we shall substitute the coordinates of F into the equation of the line. When $x = -4$,

$$y = \frac{5}{2}(-4) + 1$$
$$= -9$$

Since the x and y coordinates match with F, the line passes through F

□

(ii)

$\triangle ADF$ is an **isosceles triangle**

(b) By inspection,

$$B\boldsymbol{\left(-3\frac{1}{3}, 2\frac{1}{3}\right)}$$

(c) Note that

$$\frac{\text{Area of } ABCD}{\text{Area of } \triangle ADF} = \frac{\text{base} \times h_1}{\frac{1}{2}(\text{base})(h_2)}$$
$$= \frac{h_1}{\frac{1}{2}(h_2)}$$
$$= \frac{1}{\frac{1}{2}(3)}$$
$$= \frac{2}{3}$$

Hence,

$$\text{Area of } ABCD = \frac{2}{3} \times 87$$
$$= \textbf{58 units}^{\boldsymbol{2}}$$

4. (a)

$$A(6,6) \qquad B(x,y) \qquad C(0,y)$$

$$\begin{aligned} AB &= BC \\ \sqrt{(6-x)^2+(6-y)^2} &= \sqrt{20} \\ (6-x)^2+(6-y)^2 &= 20 \text{ ......(1)} \end{aligned}$$

Using the equation of AB,

$$\begin{aligned} y+2x &= 18 \\ y &= 18-2x \text{ ......(2)} \end{aligned}$$

Substitute Equation (2) into Equation (1),

$$\begin{aligned} 36-12x+x^2+[6-(18-2x)]^2 &= 20 \\ 36-12x+x^2+4x^2-48x+144 &= 20 \\ 5x^2-60x+160 &= 0 \\ (x-8)(x-4) &= 0 \end{aligned}$$

$$\begin{array}{lll} x=4 & \text{or} & x=8 \\ y=10 & \text{or} & y=2 \\ (4,10) & \text{or} & (8,2) \quad \text{(N.A.)} \end{array}$$

Hence,

$$A(6,6) \qquad B(4,10) \qquad \boldsymbol{C(0,10)}$$

$$\begin{aligned} \text{Gradient of } BC &= \frac{10-10}{4-0} \\ &= 0 \end{aligned}$$

$$\therefore \boldsymbol{y = 10}$$

(b)

$$\begin{aligned} \text{Midpoint of } BC &= \left(\frac{6+0}{2}, \frac{10+6}{2}\right) \\ &= \mathbf{(3,8)} \end{aligned}$$

(c)

Shown from part (a)

(d)

$$\begin{aligned} \text{Area of parallelogram } ABCD &= \frac{1}{2}\begin{vmatrix} 0 & 3 & 4 & 0 & 0 \\ 0 & 8 & 10 & 10 & 0 \end{vmatrix} \\ &= \frac{1}{2}\left|70-32\right| \\ &= \mathbf{19 \text{ units}^2} \end{aligned}$$

5. (a)

$$\text{Gradient of } AB = 2$$
$$\therefore \text{Gradient of } l_1 = -\frac{1}{2}$$

Hence, substituting $P(2,3)$,

$$y - 3 = -\frac{1}{2}(x-2)$$
$$\therefore \boldsymbol{y = -\frac{1}{2}x + 4}$$

(b) Substitute $x = 4$ into the equation of l_1,

$$\begin{aligned} y &= -\frac{1}{2}(4) + 4 \\ &= 2 \end{aligned}$$

Hence, $(4,2)$ is a point of the line **(shown)**

□

(c) Let the coordinates be $D(x,y)$

$$\text{Midpoint of } AB = \left(\frac{4+x}{2}, \frac{2+y}{2}\right) = (2,3)$$

Hence,

$$\therefore \boldsymbol{D(0,4)}$$

(d)

$$\begin{aligned} \text{Length of } CP &= \sqrt{(4-2)^2 + (2-3)^2} \\ &= \sqrt{5} \end{aligned}$$

$$\sqrt{(x-2)^2 + (y-3)^2} = \sqrt{5} \text{ ......(1)}$$

Since A lies on the line $y + 1 = 2x$,

$$y = 2x - 1 \text{ ......(2)}$$

Substitute Equation (2) into Equation (1),

$$\begin{aligned} \sqrt{(x-2)^2 + (2x-1-3)^2} &= \sqrt{5} \\ x^2 - 4x + 4 + 4x^2 - 16x + 16 &= 5 \\ 5x^2 - 20x + 15 &= 0 \\ (x-3)(x-1) &= 0 \end{aligned}$$

$$\therefore x = 3 \qquad \text{or} \qquad x = 1 \text{ (rej)}$$

Substitute $x = 3$ into Equation (2),

$$\begin{aligned} y &= 2(3) - 1 \\ &= 5 \end{aligned}$$

$$\therefore \boldsymbol{A(3,5)}$$

(e)

$$\begin{aligned} \text{Area of parallelogram } ABCD &= \frac{1}{2}\begin{vmatrix} 3 & 0 & 0 & 4 & 3 \\ 5 & 4 & -1 & 2 & 5 \end{vmatrix} \\ &= \frac{1}{2}\left|32 - 2\right| \\ &= \textbf{15 units}^{\mathbf{2}} \end{aligned}$$

9 Further Coordinate Geometry

9.1 Full Solutions

1. (a) Let the centre be $C(-1, b)$

$$\frac{6-b}{3+1} = \frac{3}{4}$$
$$b = 3$$
$$\therefore C(-1, 3)$$

$$\text{Radius} = \sqrt{(-1-3)^2 + (3-6)^2} = 5 \text{ units}$$

Hence, the equation of the circle C is

$$(x+1)^2 + (y-3)^2 = 25$$
$$x^2 + y^2 + 2x - 6y - 15 = 0 \textbf{ (shown)}$$

□

(b) When $y = 0$,

$$(x+1)^2 + 9 = 25$$
$$x = 3 \qquad \text{or} \qquad x = -5$$

Since the circle meets the x-axis at 2 distinct points, the x-axis is **not tangent**

(c)

$$\text{Shortest distance} = \sqrt{(5)^2 - (1)^2} = \sqrt{24} = \mathbf{4.90 \text{ units}}$$

2. (a) Let the x-coordinates of the centre of the circle be a

$$(17-a)^2 = (a-1)^2 + 8^2$$
$$289 - 34a + a^2 = a^2 - 2a + 1 + 64$$
$$224 = 32a$$
$$a = 7$$

$$\text{Radius} = 17 - 7 = 10 \text{ units } \textbf{(shown)}$$

□

(b)

$$\text{Centre} = \mathbf{(7, 1)}$$

(c)

$$(x-7)^2 + (y-1)^2 = 10^2$$
$$\boldsymbol{x^2 + y^2 - 14x - 2y - 50 = 0}$$

(d)

$$\text{Centre of reflected circle} = (7, -3)$$

$$\text{Distance} = \sqrt{(3-7)^2 + (10+3)^2} = \sqrt{185} = 13.601... > 0 \textbf{ (shown)}$$

□

3. (a)

$$3x^2 - 30x + 75 - 12y + 3y^2 = 0$$
$$x^2 + y^2 - 10x - 4y + 25 = 0$$
$$\text{Centre} = \mathbf{(5, 2)}$$
$$\text{Radius} = \sqrt{(5)^2 + (2)^2 - 25}$$
$$= \mathbf{2\ units}$$

(b) Since the y-coordinate of the centre of C_1 is 2 and radius of the circle is also 2 units, thus the circle C_1 touches the x-axis

(c)

$$\text{Centre} = (5, 2)$$
$$\text{Radius} = \sqrt{(5-1)^2 + (2-6)^2}$$
$$= 4\sqrt{2} \text{ units}$$

Hence, the equation of the circle C_2 is

$$(x-5)^2 + (y-2)^2 = \left(4\sqrt{2}\right)^2$$
$$(x-5)^2 + (y-2)^2 = 32$$
$$\boldsymbol{x^2 + y^2 - 10x - 4y - 2 = 0}$$

(d)

$$\text{Radius of } C_2 = 4\sqrt{2}$$

Let $B(x, y)$

$$\left(\frac{x+1}{2}, \frac{y+6}{2}\right) = (5, -2)$$
$$\therefore x = 9 \qquad \text{or} \qquad y = -2$$
$$\text{Gradient of line} = \frac{4}{-4}$$
$$= -1$$
$$\text{Gradient of tangent at } B = 1$$

Hence, the equation of the tangent is

$$y - (-2) = (x - 9)$$
$$y = x - 11$$

(e) Let $P(x, 6)$,

$$(x-5)^2 + (6-2)^2 = 32$$
$$(x-5)^2 = 32 - 16$$
$$x = 9 \qquad \text{or} \qquad x = 1 \text{ (N.A)}$$
$$\therefore \boldsymbol{x = 9}$$

4. (a)

$$\text{Radius} = 5 \text{ units}$$
$$\text{Centre} = (-2, 0)$$
$$\therefore \boldsymbol{(x+2)^2 + y^2 = 25}$$

(b) The centre has changed to

$$\text{Centre} = (0, 2)$$
$$\therefore \boldsymbol{x^2 + (y-2)^2 = 25}$$

5. (a) At the x-intercept, $y = 0$

$$\therefore Q(2,0)$$

$$\begin{aligned}\text{Radius} &= \sqrt{(2)^2+(2)^2}\\ &= \sqrt{8}\end{aligned}$$

Hence, the equation of the circle C_1 is

$$(x-2)^2+y^2=8$$

$$\therefore \boldsymbol{x^2+y^2-4x-4=0}$$

(b) Q is the midpoint of AP. Let $P(x,y)$

$$\left(\frac{x+0}{2},\frac{y+2}{2}\right)=(2,0)$$

$$P(4,-2)$$

$$\begin{aligned}\text{Radius} &= 2AQ\\ &= 2\sqrt{8}\end{aligned}$$

Hence, the equation of the circle C_2 is

$$(x-4)^2+(y+2)^2=4(8)$$

$$\therefore \boldsymbol{x^2+y^2-8x+4y-12=0}$$

(c) Substitute $B(k,0)$,

$$\begin{aligned}k^2+(0)^2-8(k)+4(0)-12&=0\\ k^2-8k-12&=0\end{aligned}$$

$$\begin{aligned}k &= \frac{-(-8)\pm\sqrt{(-8)^2-4(1)(-12)}}{2(1)}\\ &= \frac{8\pm\sqrt{112}}{2}\\ &= \frac{8\pm4\sqrt{17}}{2}\ \text{(rej -ve)}\end{aligned}$$

$$\therefore \boldsymbol{k=4+2\sqrt{7}}$$

(d)

$$\begin{aligned}\text{Gradient of radius} &= \frac{0+2}{4+2\sqrt{7}-4}\\ &= \frac{1}{\sqrt{7}}\end{aligned}$$

$$\begin{aligned}\text{Gradient of tangent} &= \frac{-1}{\left(\frac{1}{\sqrt{7}}\right)}\\ &= \boldsymbol{-\sqrt{7}\ \text{(shown)}}\end{aligned}$$

At the y-axis,

$$0=-\sqrt{7}\left(4+2\sqrt{7}\right)+c$$

$$c=\boldsymbol{4\sqrt{7}+14\ \text{(shown)}}$$

□

10 Linear Law

10.1 Full Solutions

1.

$$y = \frac{x}{b\sqrt{x} - a}$$
$$\frac{y}{x} = b\sqrt{x} - a$$

Let

$$Y = \frac{x}{y} \qquad X = \sqrt{x}$$
$$Y = bX - a$$

To find the gradient,

$$b = \frac{11 - 3}{3 - 5}$$
$$= -4$$

When $X = 5$ and $Y = 3$,

$$3 = -4(5) - a$$
$$a = -23$$

$$\therefore \boldsymbol{a = -23} \qquad \boldsymbol{b = -4}$$

2. (a)

$$\text{Gradient} = \frac{12 - 8}{2 - 3}$$
$$= -4$$

$$\lg y = -4x^2 + c$$

Substitute $(3, 8)$,

$$8 = -4(3) + c$$
$$c = 20$$

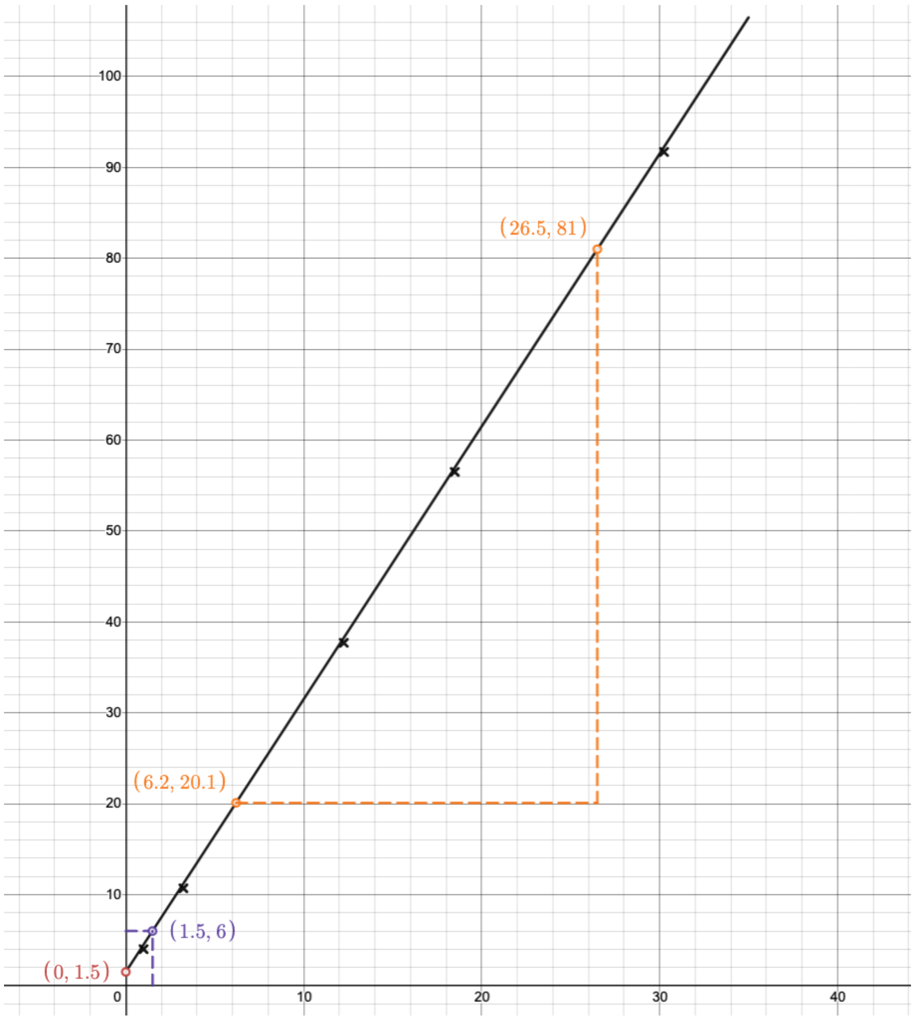
$$\lg y = -4x^2 + 20$$
$$\boldsymbol{y = 10^{-4x^2+20}}$$

(b) (i)

$$\sqrt{y} = a\left(x^2 + b\right)$$
$$\sqrt{y} = ax^2 + ab$$

Hence, we are plotting a graph of $\sqrt{y}$ against x^2

x^2	1	3.24	12.25	18.49	30.25
$\sqrt{y}$	4	10.7	37.7	56.5	91.7



(ii) From the graph,

$$a = \frac{81 - 20.1}{26.5 - 6.2}$$
$$= \mathbf{3}$$

$$ab = 1.5$$
$$b = \mathbf{\frac{1}{2}}$$

(iii) From the graph, when $y = 36$, $\sqrt{y} = 6$

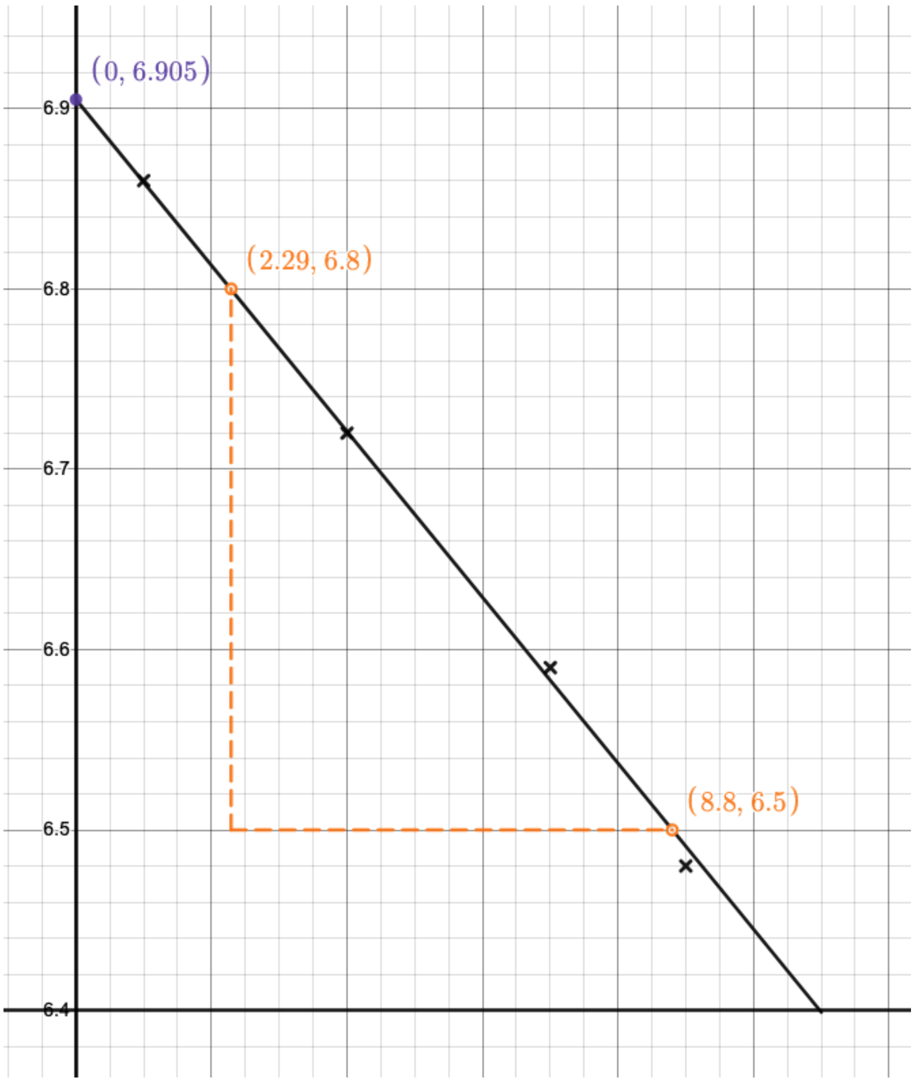
$$x = \mathbf{1.5}$$

3. (a)

$$V = V_0 e^{kt}$$
$$V = kt + \ln V_0$$

Plot a graph of V against t

t	1	4	7	9
$\ln V$	6.86	6.72	6.59	6.48



(b) From the graph,

$$\ln V_0 = 6.905$$
$$V_0 = e^{6.905}$$
$$= 997.2485...$$
$$= \mathbf{997\ (3.s.f.)}$$

V_0 represents the **initial starting price of the mobile phone**

(c) From the graph,

$$\begin{aligned} k &= \frac{6.8-6.5}{2.29-8.8} \\ &= -\frac{10}{217} \\ &= -0.046082... \\ &= \mathbf{-0.0461\ (3.s.f.)} \end{aligned}$$

(d) Assuming that the model is appropriate, substitute the values of V_0 and k in

$$\begin{aligned} V &= \left(e^{6.905}\right) e^{-\frac{10}{217}(15)} \\ &= 499.574012... \\ &= \mathbf{\$500\ (3.s.f.)} \end{aligned}$$

4. (a)

$$\begin{aligned} e^y - 1 &= \frac{1.6-1}{0.5-0.2}\left(x^2-0.2\right) \\ e^y - 1 &= 2\left(x^2-0.2\right) \\ e^y &= 2x^2+0.6 \end{aligned}$$

Hence, when $x = 0$,

$$\therefore e^y = \mathbf{0.6}$$

(b)

$$\ln e^y = \ln\left(2x^2+0.6\right)$$

$$\therefore \boldsymbol{y = \ln\left(2x^2+0.6\right)}$$

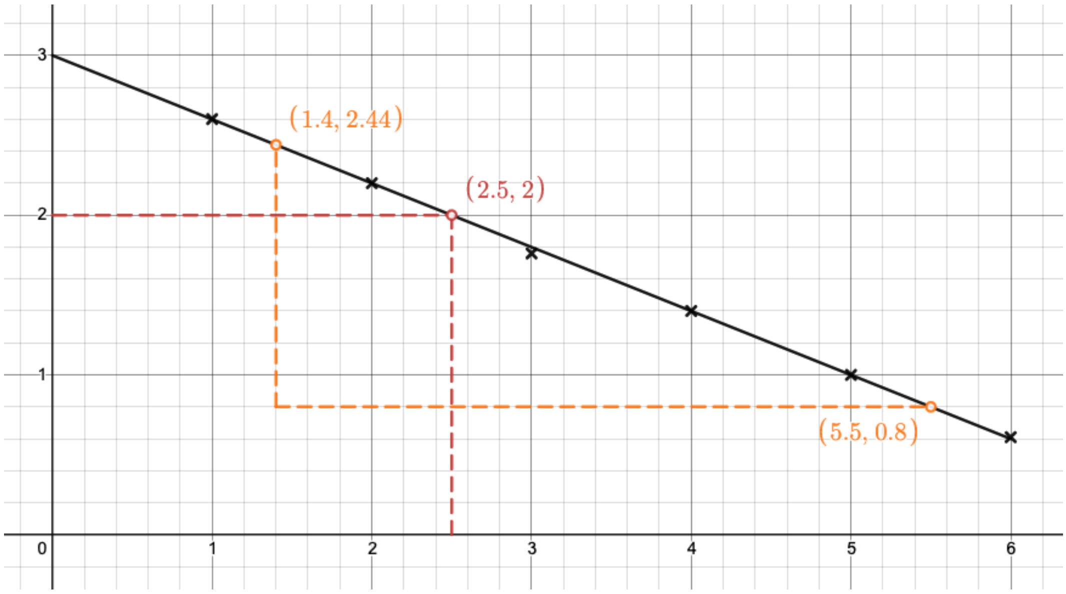
5. (a) Table

x^2y	2.601	2.20	1.75	1.42	1.00	0.61

(b)

$$y = \frac{h}{kx} + \frac{1}{kx^2}$$
$$x^2y = \frac{h}{k}x + \frac{1}{k}$$

Hence, we are plotting x^2y against x



(c) (i) When $x = 2.5$,

$$x^2y = 2$$
$$y = \frac{2}{(2.5)^2}$$
$$= \mathbf{0.32}$$

(ii) From the graph,

$$\frac{1}{k} = 3$$
$$k = \mathbf{\frac{1}{3}}$$

(iii) From the graph,

$$\frac{h}{k} = \frac{2.44 - 0.8}{1.4 - 5.5}$$
$$h = \mathbf{-\frac{2}{15}}$$

11 Proofs of Plane Geometry

11.1 Full Solutions

1. (a) (i)

$$\angle ACF = \angle FGC \text{ (alternate segment theorem)}$$
$$\angle ACF = \angle EFC \text{ (alternate angles)}$$
$$\therefore \angle FGC = \angle EFC \text{ (A)}$$
$$\angle EFC = \angle FCG \text{ (common angles) (A)}$$

By the AA similarity test, $\triangle ECF$ and $\triangle FCG$ are similar

(ii) From part (a)(i),

$$\frac{EC}{FC} = \frac{CF}{CG}$$

Hence,

$$\boldsymbol{EC \times CG = (CF)^2}$$

□

(b)

$$\angle GEF = \angle HEC \text{ (vertically opposite angles) (A)}$$
$$\angle FGE = \angle CHE \text{ (angles in the same segment) (A)}$$

By the AA similarity test, $\triangle FGE$ and $\triangle CHE$ are similar

From the similar triangles,

$$\frac{FE}{EC} = \frac{EG}{EH}$$

$$\begin{aligned}(FE)(EH) &= (EG)(EC)\\ &= (CG - EC)(EC)\\ &= (CG)(EC) - (EC)^2\\ &= CF^2 - EC^2 \textbf{ (shown)}\end{aligned}$$

□

2. (a)

$$\angle ABP = \angle APQ \text{ (alternate segment theorem)}$$

Since PA bisects $\angle QPB$,

$$\angle APQ = \angle APB$$
$$\therefore \angle ABP = \angle APB \text{ (angles of an isosceles triangle } APB\text{)}$$

Hence,

$$AP = AB \textbf{ (shown)}$$

□

(b)

$$\angle ACB = \angle APB \text{ (angles in the same segment)}$$
$$\angle ACP = \angle ABP = \angle APB \text{ (angles in the same segment)}$$
$$\therefore \angle ACB = \angle ACP$$

Hence,

CD bisects $\angle PCB$ (shown)

□

(c)

$$\angle ACB = \angle ACP \text{ (from part (b))}$$
$$\angle CPD = \angle CAB \text{ (angles in the same segment)}$$

Hence,

$\triangle CDX$ and $\triangle CBA$ are similar

□

3. (a)

$$\angle BCA = \angle ACE \text{ (common angles) (A)}$$

$$\begin{aligned}\angle ABC &= \angle CAY \text{ (alternate segment theorem)}\\ &= \angle EAC \text{ (AC bisects $\angle DAY$) (A)}\end{aligned}$$

$\therefore$ By the AA similarity test, $\triangle BAC$ and $\triangle AEC$ are similar

$$\begin{aligned}\frac{AC}{EC} &= \frac{BC}{AC}\\ AC^2 &= EC \times BC \textbf{ (shown)}\end{aligned}$$

□

(b)

$$\angle CAY = \angle EAC \text{ (AC bisects $\angle DAY$)}$$

$$\angle BAX = \angle EAB \text{ (AB bisects $\angle BAX$)}$$

$$\begin{aligned}\angle BAX + \angle EAB + \angle EAC + \angle CAY &= 180^\circ \text{ (angles on a straight line)}\\ 2\angle EAB + 2\angle EAC &= 180^\circ\\ \angle EAB + \angle EAC = \angle BAC &= 90^\circ\end{aligned}$$

Since $\angle BAC = 90^\circ$, BC is a diameter of the circle (shown)

□

(c)

$$\angle ABE = \angle CAY \text{ (alternate segment theorem)}$$

$$\angle CAY = \angle EAC \text{ (AC bisects $\angle BAY$)}$$

$$\therefore \angle ABE = \angle EAC$$

$$\angle EAB + \angle EAC = \angle EAB + \angle ABE = 90^\circ \text{ (from part (b))}$$

$$\angle AEB = 90^\circ \text{ (angles in a triangle)}$$

$\therefore$ **Hence, AD and BC are perpendicular**

□

4. (a)

$$\angle ADB = 90^\circ \text{ (angles in a semicircle)}$$

$$\angle AEO = \angle CED \text{ (vertically opposite angles) (A)}$$

$$\begin{aligned}\angle EAO &= 90^\circ - \angle AEP \text{ (angles in a triangle)}\\ &= 90^\circ - \angle CED\\ &= \angle ECD \text{ (A)}\end{aligned}$$

By the AA similarity test, $\triangle AEO$ is similar to $\triangle CED$

$$\frac{AE}{CE} = \frac{EO}{ED} = \frac{AO}{CD}$$

$$\therefore \boldsymbol{AE \times ED = OE \times EC} \textbf{ (shown)}$$

□

(b) OG is perpendicular to AB (given) and OG passes through the centre. Hence, it is equidistant from A and B. All points along OG will be equidistant from A and B. Since C extends from OG, C will be equidistant from A and B **(shown)**

□

(c)

$$\angle COB = 90^\circ \textbf{ (given)}$$

Using angles in a semicircle, there is a circle, with CB as its diameter that passes through the point O **(shown)**

□

5. (a)

$$DT \text{ is parallel to } AB \text{ (midpoint theorem)}$$

$$\begin{aligned}\angle AFD &= \angle TDF \text{ (alternate angles)}\\ &= \angle FED \text{ (alternate segment theorem)}\end{aligned}$$

Hence, $\boldsymbol{AB}$ is a tangent at $\boldsymbol{F}$ (shown)

□

(b)

$$\angle TDF = \angle DCF \text{ (angles in an isosceles triangle) (A)}$$

$$\angle DFE \text{ is a common angle (A)}$$

$$\angle DCF = \angle DEF \text{ (angles in the same segment) (A)}$$

By the AAA similarity test, $\triangle DFT$ is similar to $\triangle EFD$

$$\begin{aligned}\frac{DF}{EF} &= \frac{FT}{FD}\\ DF^2 &= FT \times EF\\ &= FT \times (ET + TF)\\ &= FT^2 + FT \times ET\end{aligned}$$

$$\therefore \boldsymbol{DF^2 - FT^2 = FT \times ET} \textbf{ (shown)}$$

□

12 Differentiation

12.1 Full Solutions

1. (a) By Pythagoras' Theorem,

$$\begin{aligned}\left(\frac{h}{2}\right)^2 + r^2 &= 35^2\\ \frac{h^2}{4} &= 1225 - r^2\\ h^2 &= 4\left(1225 - r^2\right)\\ h &= 2\sqrt{1225 - r^2}\ \textbf{(shown)}\end{aligned}$$

□

(b) Volume of the cylinder can be computed as

$$\begin{aligned}V &= \pi r^2\left(2\sqrt{1225 - r^2}\right)\\ &= 2\pi r^2\left(1225 - r^2\right)^{\frac{1}{2}}\end{aligned}$$

Hence, using the product rule,

$$\begin{aligned}\frac{dV}{dr} &= 2\pi r^2\left[\frac{1}{2}(-2r)\left(1225 - r^2\right)^{-\frac{1}{2}}\right] + \left(1225 - r^2\right)^{\frac{1}{2}}(4\pi r)\\ &= \frac{-2\pi r^3}{\sqrt{1225 - r^2}} + 4\pi r\sqrt{1225 - r^2}\end{aligned}$$

Since the volume of the cylinder is maximum,

$$\frac{-2\pi r^3}{\sqrt{1225 - r^2}} + 4\pi r\sqrt{1225 - r^2} = 0$$

$$\begin{aligned}r^3 &= 2r\left(1225 - r^2\right)\\ 3r^3 &= 2450r\\ r &= \sqrt{816\frac{2}{3}} \quad \text{(rej 0 and -ve)}\end{aligned}$$

$$\begin{aligned}\text{Maximum volume} &= \pi\left(\sqrt{816\frac{2}{3}}\right)^2\left[2\sqrt{1225 - 816\frac{2}{3}}\right]\\ &= 103688.8637...\\ &= \mathbf{104000\ cm^3\ (3.s.f.)}\end{aligned}$$

x	$\sqrt{816\frac{2}{3}}\ (-)$	$\sqrt{816\frac{2}{3}}$	$\sqrt{816\frac{2}{3}}\ (+)$
$\frac{dy}{dx}$	+ve	0	-ve

Hence, *V* is maximum

2. (a)

$$\begin{aligned}\frac{d}{dx}(\sec x) &= \frac{d}{dx}\left(\frac{1}{\cos x}\right)\\ &= \frac{(\cos x)(0)-(1)(-\sin x)}{\cos^2 x}\\ &= \frac{\sin x}{\cos^2 x}\\ &= \sec x \tan x \ \textbf{(shown)}\end{aligned}$$

□

(b)

$$\begin{aligned}\frac{dy}{dx} &= 1-\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}\\ &= 1-\frac{\sec x(\tan x+\sec x)}{\sec x+\tan x}\\ &= \mathbf{1-\sec x}\end{aligned}$$

(c)

$$\begin{aligned}\frac{dy}{dx} &= 1-\sec x\\ &= 1-\frac{1}{\cos x}\\ &= \frac{\cos x-1}{\cos x}\end{aligned}$$

Note that the principal domain of $\cos x$ is $(-1, 1)$. With the given range in the question,

$$0 < \cos x < 1$$

Note that the numerator of $\frac{dy}{dx}$ will always be negative, and the denominator of $\frac{dy}{dx}$ will always be positive. Hence

$$\boldsymbol{\frac{dy}{dx} < 0, \quad \textbf{decreasing function}}$$

□

3. (a) (i) Using similar triangles,

$$\begin{aligned}\frac{28-h}{28} &= \frac{r}{10}\\ 28-h &= \frac{28}{10}r\\ h &= 28-\frac{14}{5}r \textbf{ (shown)}\end{aligned}$$

□

(ii)

$$\begin{aligned}\text{Volume of cylinder} &= \pi r^2\left(28-\frac{14}{5}r\right)\\ &= 14\pi r^2\left(2-\frac{1}{5}r\right) \textbf{ (shown)}\end{aligned}$$

□

(b) (i)

$$\begin{aligned}\frac{dV}{dr} &= 56\pi r-\frac{14}{5}\pi\left(3r^2\right)\\ &= 14\pi r\left(4-\frac{3}{5}r\right)\end{aligned}$$

Given that the volume of maximum,

$$14\pi r\left(4-\frac{3}{5}r\right)=0$$

$$r=0 \text{ (rej)} \qquad \text{or} \qquad r=6\frac{2}{3}$$

$$\begin{aligned}\frac{d^2V}{dr^2} &= 56\pi-\frac{84}{5}\pi r\\ &= 56\pi-\frac{84}{5}\pi\left(6\frac{2}{3}\right)\\ &= -175.93... < 0\end{aligned}$$

$$\text{Since } \frac{d^2V}{dr^2}<0,\ V \text{ is maximum}$$

$$\begin{aligned}\text{Max volume} &= 14\pi\left(\frac{20}{3}\right)^2\left[4-\frac{3}{5}\left(\frac{20}{3}\right)\right]\\ &= \mathbf{414\frac{22}{27}\ cm^3}\end{aligned}$$

(ii)

$$\begin{aligned}\text{Volume of cone} &= \frac{1}{3}\pi(10)^2(28)\\ &= \frac{2800}{3}\pi\text{ cm}^3\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\text{Volume of cylinder}}{\text{Volume of cone}} &= \frac{11200\pi}{27}\times\frac{3}{2800\pi}\\ &= \frac{4}{9} \textbf{ (shown)}\end{aligned}$$

□

4.

$$\begin{aligned}\frac{dy}{dx} &= x^2\left(-2e^{1-2x}\right) + e^{1-2x}(2x)\\ &= -2x^2e^{1-2x} + 2xe^{1-2x}\\ &= -2y + \frac{2y}{x}\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= -2\left(\frac{dy}{dx}\right) + 2x\left(-2e^{1-2x}\right) + 2e^{1-2x}\\ &= -2\left(\frac{dy}{dx}\right) - 4xe^{1-2x} + 2e^{1-2x}\\ &= -2\left(\frac{dy}{dx}\right) - \frac{4y}{x} + \frac{2y}{x^2}\end{aligned}$$

$$\begin{aligned}\therefore \frac{d^2y}{dx^2} - \frac{2y}{x^2} &= -2\left(\frac{dy}{dx}\right) - \frac{4y}{x}\\ &= -2\left(\frac{dy}{dx}\right) - 2\left(\frac{dy}{dx} + 2y\right)\\ &= -4\left(\frac{dy}{dx}\right) - 4y\\ &= -4\left(\frac{dy}{dx} + y\right)\end{aligned}$$

$$\therefore \boldsymbol{k = -4}$$

5. (a) Let $AC = r$ and $BC = h$

$$r^2 = 16 - h^2$$

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi\left(16 - h^2\right) h \\ &= \frac{16}{3}\pi h - \frac{1}{3}\pi h^3 \end{aligned}$$

$$\frac{dV}{dh} = \frac{16}{3}\pi - \pi h^2$$

Since maximum, $\frac{dV}{dh} = 0$

$$\begin{aligned} \frac{16}{3}\pi &= \pi h^2 \\ h &= \frac{4}{\sqrt{3}} \quad \text{(rej -ve)} \end{aligned}$$

$$\begin{aligned} \frac{d^2V}{dh^2} &= -2\pi h \\ &= -\frac{8}{\sqrt{3}}\pi < 0 \end{aligned}$$

Hence, V is maximum

$$\boldsymbol{h = \frac{4}{\sqrt{3}}} \textbf{ cm}$$

(b)

$$\begin{aligned} r^2 &= 16 - \left(\frac{4}{\sqrt{3}}\right)^2 \\ r &= \frac{4\sqrt{2}}{\sqrt{3}} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{h}{r} &= \frac{\left(\frac{4}{\sqrt{3}}\right)}{\left(\frac{4\sqrt{2}}{\sqrt{3}}\right)} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$BC : CA = 1 : \sqrt{2} \textbf{ (shown)}$$

□

13 Integration

13.1 Full Solutions

1. (a)

$$\begin{aligned}
\text{LHS} &= \frac{2}{\tan\theta + \cot\theta} \\
&= 2 \div \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) \\
&= 2 \div \left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right) \\
&= 2 \div \left(\frac{1}{\cos\theta\sin\theta}\right) \\
&= 2\sin\theta\cos\theta \\
&= \sin 2\theta \\
&= \text{RHS } \textbf{(shown)}
\end{aligned}$$

□

(b)

$$\begin{aligned}
\int_0^p \frac{4}{\tan 2x + \cot 2x}\ dx &= 2\int_0^p \sin 4x\ dx \\
&= 2\left[-\frac{\cos 4x}{4}\right]_0^p \\
&= \left(-\frac{1}{2}\cos 4p\right) - \left(-\frac{1}{2}\cos 0\right) \\
&= -\frac{1}{2}\cos 4p + \frac{1}{2}
\end{aligned}$$

Hence,

$$\begin{aligned}
-\frac{1}{2}\cos 4p + \frac{1}{2} &= \frac{1}{4} \\
-\frac{1}{2}\cos 4p &= -\frac{1}{4} \\
\cos 4p &= \frac{1}{2} \\
4p &= \frac{\pi}{3} \\
p &= \boldsymbol{\frac{\pi}{12}}
\end{aligned}$$

2. (a) At minimum gradient, $\frac{d^2y}{dx^2} = 0$

$$a\left(\frac{1}{3}\right) - 2 = 0$$
$$\frac{a}{3} = 2$$
$$a = 6 \textbf{ (shown)}$$

□

(b)

$$\frac{dy}{dx} = \int (6x - 2)\ dx$$
$$= 3x^2 - 2x + c \quad \text{where } c \text{ is an arbitrary constant}$$

Since the tangent of the curve at the point $(1, 4)$ is $y = 2x + 2$, the gradient of the tangent is 2

$$3(1)^2 - 2(1) + c = 2$$
$$c = 1$$

$$y = \int \left(3x^2 - 2x + 1\right)\ dx$$
$$= x^3 - x^2 + x + d \quad \text{where } d \text{ is an arbitrary constant}$$

Substituting $(1, 4)$,

$$4 = (1)^3 - (1)^2 + 1 + d$$
$$d = 3$$

$$\therefore \boldsymbol{y = x^3 - x^2 + x + 3}$$

3. (a)

$$\begin{aligned}\int_0^{\frac{\pi}{8}} f'(x)\ dx &= \frac{\pi}{16} - \frac{1}{8}\\ \frac{\left(\frac{\pi}{8}\right)}{2} - \frac{\left[\sin k\left(\frac{\pi}{8}\right)\right]}{8} &= \frac{\pi}{16} - \frac{1}{8}\\ \sin\left(\frac{k\pi}{8}\right) &= 1\\ \frac{k\pi}{8} &= \frac{\pi}{2}\\ k &= 4 \ \textbf{(shown)}\end{aligned}$$

□

(b)

$$\begin{aligned}\int f'(x)\ dx &= \frac{x}{2} - \frac{\sin 4x}{8} + c\\ f'(x) &= \frac{1}{2} - \frac{1}{8}(4\cos 4x)\\ &= \frac{1}{2} - \frac{1}{2}\cos 4x\\ &= \frac{1}{2} - \frac{1}{2}\left(1 - 2\sin^2 2x\right)\\ &= \boldsymbol{\sin^2 2x}\end{aligned}$$

(c)

$$\int f'(x) = f(x) = \frac{x}{2} - \frac{\sin 4x}{8} + c$$

At $\left(\frac{\pi}{4}, 0\right)$,

$$\begin{aligned}0 &= \frac{\pi}{8} - 0 + c\\ c &= -\frac{\pi}{8}\end{aligned}$$

$$\therefore \boldsymbol{f(x) = \frac{x}{2} - \frac{\sin 4x}{8} - \frac{\pi}{8}}$$

4. (a)

$$f'(x) = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$
$$f(x) = \int \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\ dx$$
$$= \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$$

At $(4, 0)$,

$$\frac{2}{3}(4)^{\frac{3}{2}} - 2(4)^{\frac{1}{2}} + c = 0$$
$$c = -\frac{4}{3}$$
$$\therefore f(x) = \boldsymbol{\frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - \frac{4}{3}}$$

(b) At Q,

$$f'(4) = \left.\frac{dy}{dx}\right|_{x=4}$$
$$= 4^{\frac{1}{2}} - 4^{-\frac{1}{2}}$$

Hence, the equation of PQ,

$$y = \frac{3}{2}(x-4)$$
$$y = \frac{3}{2}x - 6$$

By observing the equation, at P,

$$\boldsymbol{y = -6}$$

(c)

$$\text{Area of shaded region} = \frac{1}{2}(4)(6) - \left|\int_0^4 \left(\frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - \frac{4}{3}\right)\ dx\right|$$
$$= 12 + \left[\frac{4}{15}x^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} - \frac{4}{3}x\right]_0^4$$
$$= 12 + \left[\frac{4}{15}(4)^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} - \frac{4}{3}(4)\right]$$
$$= 12 - \frac{112}{15}$$
$$= \boldsymbol{4\frac{8}{15}\ \textbf{units}^2}$$

5. (a)

$$\begin{aligned}\sin(A+B)+\sin(A-B) &= \sin A\cos B+\sin B\cos A+\sin A\cos B-\sin B\cos A\\ &= 2\sin A\cos B\end{aligned}$$

$$\therefore \boldsymbol{k=2}$$

(b)

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sin 2x\cos x\ dx &= \frac{1}{2}\int_0^{\frac{\pi}{4}} 2\sin 2x\cos x\ dx\\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}} [\sin(2x+x)+\sin(2x-x)]\ dx\\ &= \frac{1}{2}\int_0^{\frac{\pi}{4}} [\sin 3x+\sin x]\ dx\\ &= \frac{1}{2}\left[-\frac{1}{3}\cos 3x-\cos x\right]_0^{\frac{\pi}{4}}\\ &= \frac{1}{2}\left\{\left[-\left(\frac{1}{3}\right)\left(\frac{1}{\sqrt{2}}\right)-\frac{1}{\sqrt{2}}\right]-\left[-\frac{1}{3}(1)-(1)\right]\right\}\\ &= \frac{1}{2}\left[-\frac{1}{3\sqrt{2}}-\frac{1}{\sqrt{2}}+\frac{4}{3}\right]\\ &= \frac{1}{2}\left(\frac{-1-3+4\sqrt{2}}{3\sqrt{2}}\right)\\ &= \frac{2\sqrt{2}-2}{3\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\\ &= \boldsymbol{\frac{4-\sqrt{2}}{6}}\end{aligned}$$

14 Differentiation & Integration

14.1 Full Solutions

1. (a)

$$\begin{aligned}\frac{d}{dx}\left[(x-5)\sqrt{2x-1}\right] &= \sqrt{2x-1}+(x-5)\left[\frac{1}{2}(2x-1)^{-\frac{1}{2}}(2)\right]\\ &= \sqrt{2x-1}+\frac{x-5}{\sqrt{2x-1}}\\ &= \frac{2x-1+x-5}{\sqrt{2x-1}}\\ &= \boldsymbol{\frac{3x-6}{\sqrt{2x-1}}}\end{aligned}$$

(b)

$$\begin{aligned}\int_1^2 \frac{3x-9}{\sqrt{2x-1}}\,dx &= \int_1^2 \left[\frac{3x-6}{\sqrt{2x-1}}-\frac{3}{\sqrt{2x-1}}\right]\,dx\\ &= \int_1^2 \frac{3x-6}{\sqrt{2x-1}}\,dx - \int_1^2 \frac{3}{\sqrt{2x-1}}\,dx\\ &= \left[(x-5)\sqrt{2x-1}\right]_1^2 - \left[\frac{3(2x-1)^{\frac{1}{2}}}{2\left(\frac{1}{2}\right)}\right]_1^2\\ &= \left[-3\sqrt{3}-(-4)\right]-\left[3\sqrt{3}-3\right]\\ &= \boldsymbol{7-6\sqrt{3}}\end{aligned}$$

2. (a)

$$\begin{aligned}\frac{d}{dx}(\sin x\cos x) &= \sin x(-\sin x)+\cos x(\cos x)\\ &= \cos^2 x-\sin^2 x\\ &= \cos^2 x-\left(1-\cos^2 x\right)\\ &= 2\cos^2 x-1 \ \textbf{(shown)}\end{aligned}$$

□

(b)

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \cos^2 x\,dx &= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(2\cos^2 x-1\right)+1\,dx\\ &= \frac{1}{2}\left\{\left[\sin x\cos x\right]_0^{\frac{\pi}{4}}+\left[x\right]_0^{\frac{\pi}{4}}\right\}\\ &= \frac{1}{2}\left[\frac{1}{2}+\frac{\pi}{4}\right]\\ &= \boldsymbol{\frac{1}{4}+\frac{\pi}{8}}\end{aligned}$$

3. (a)

$$
\begin{aligned}
f'(x) &= \left(e^x + \frac{1}{e^x}\right)^2 \\
y &= \int \left(e^x + \frac{1}{e^x}\right)^2 \, dx \\
&= \int \left(e^{2x} + 2 + e^{-2x}\right) \, dx \\
&= -\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c
\end{aligned}
$$

At $(0, 3)$,

$$
\begin{aligned}
3 &= \frac{1}{2}e^0 + 2(0) - \frac{1}{2}e^0 + c \\
c &= 3
\end{aligned}
$$

$$\therefore \boldsymbol{y = \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + 3}$$

(b)

$$
\begin{aligned}
f'(x) &= e^{2x} + 2 + e^{-2x} \\
f''(x) &= 2e^{2x} - 2e^{-2x}
\end{aligned}
$$

Since $f''(x) = 3$,

$$2e^{2x} - 2e^{-2x} = 3$$

Let $e^{2x} = a$,

$$
\begin{aligned}
2a - \frac{2}{a} &= 3 \\
2a^2 - 3a - 2 &= 0 \\
(2a + 1)(a - 2) &= 0
\end{aligned}
$$

$$
\begin{aligned}
a = 2 \quad &\text{or} \quad a = -\frac{1}{2} \\
e^{2x} = 2 \quad &\text{or} \quad e^{2x} = -\frac{1}{2} \text{ (N.A.)}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
2x &= \ln 2 \\
x &= \boldsymbol{\ln \sqrt{2}}
\end{aligned}
$$

4. (a)

$$\begin{aligned} y &= e^x \sin x \\ \frac{dy}{dx} &= e^x \sin x + e^x \cos x \\ \frac{d^2y}{dx^2} &= e^x \sin x + e^x \cos x - e^x \sin x + e^x \cos x \\ &= 2e^x \cos x \end{aligned}$$

$$\begin{aligned} \therefore 2\left(\frac{dy}{dx}\right) - \frac{d^2y}{dx^2} &= 2\left(e^x \sin x + e^x \cos x\right) - 2e^x \cos x \\ &= 2e^x \sin x \\ &= 2y \textbf{ (shown)} \end{aligned}$$

□

(b)

$$\begin{aligned} -\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right) &= 2y \\ \therefore -\frac{dy}{dx} + 2y &= 2\int e^x \sin x \ dx \\ -e^x \sin x - e^x \cos x + 2e^x \sin x &= 2\int e^x \sin x \ dx \\ \therefore \int e^x \sin x \ dx &= \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + c \end{aligned}$$

Hence,

$$\begin{aligned} \int_0^{\frac{\pi}{3}} e^x \sin x \ dx &= \left[\frac{1}{2}\left(e^x \sin x - e^x \cos x\right)\right]_0^{\frac{\pi}{3}} \\ &= \mathbf{1.02 \ (3.s.f.)} \end{aligned}$$

5. (a)

$$\begin{aligned} y &= x^2\sqrt{2x+1} \\ \frac{dy}{dx} &= x^2\left[\frac{1}{2}(2x+1)^{-\frac{1}{2}}(2)\right] + 2x\,(2x+1)^{\frac{1}{2}} \\ &= \frac{x^2}{\sqrt{2x+1}} + 2x\sqrt{2x+1} \\ &= \frac{x^2+2x(2x+1)}{\sqrt{2x+1}} \\ &= \frac{x(5x+2)}{\sqrt{2x+1}} \ \textbf{(shown)} \end{aligned}$$

□

(b) (i) At the stationary points,

$$\frac{dy}{dx} = \frac{x(5x+2)}{\sqrt{2x+1}} = 0$$

$$x = 0 \quad \text{or} \quad x = -\frac{2}{5}$$

Stationary points are $(0,0)$ and $\left(-\frac{2}{5}, 0.0716\right)$

Using the first derivative test,

x	$0\ (-)$	0	$0\ (+)$
$\frac{dy}{dx}$	-ve	0	+ve

$\therefore (0,0)$ is a **minimum** point

x	$-0.4\ (-)$	-0.4	$0.4\ (+)$
$\frac{dy}{dx}$	+ve	0	-ve

$\therefore \left(-\frac{2}{5}, 0.0716\right)$ is a **maximum** point

(ii)

$$\begin{aligned} \int_1^5 \frac{5x^2+2x-3}{\sqrt{2x+1}}\,dx &= \int_1^5 \frac{x(5x+2)}{\sqrt{2x+1}}\,dx - 3\int_1^5 \frac{1}{\sqrt{2x+1}}\,dx \\ &= \left[x^2\sqrt{2x+1}\right]_1^5 - 3\left[\sqrt{2x+1}\right]_1^5 \\ &= \mathbf{76.4\ (3.s.f.)} \end{aligned}$$

15 Kinematics

15.1 Full Solutions

1. (a) At instantaneous rest, $v = 0$

$$2 - \frac{18}{(t+2)^2} = 0$$

$$t = 1 \quad \text{or} \quad t = -5 \text{ (N.A.)}$$

(b)

$$\begin{aligned} s &= \int \frac{dv}{dt}\, dt \\ &= 2t + \frac{18}{t+2} + c \end{aligned}$$

When $t = 1$, $s = 9$,

$$\begin{aligned} 9 &= 2(1) + \frac{18}{(1)+2} + c \\ c &= 1 \end{aligned}$$

$$s = 2t + \frac{18}{t+2} + 1$$

When $t = 0$, $s = 10$ m, when $t = 1$, $s = 9$ m and when $t = 4$, $s = 12$ m

$$\begin{aligned} \therefore \text{Total distance travelled} &= 10 - 9 + 12 - 9 \\ &= \mathbf{4\ m} \end{aligned}$$

(c) When $t = 7$,

$$\begin{aligned} v &= 2 - \frac{18}{(7+2)^2} \\ &= \frac{16}{9} \end{aligned}$$

Hence, when $t = 7$,

$$k = \mathbf{1\frac{7}{9}}$$

(d)

$$\begin{aligned} V &= -h\left(t^2 - 7t\right) + k \\ &= -ht^2 + 7ht + k \end{aligned}$$

Hence,

$$\begin{aligned} a &= \frac{dV}{dt} \\ &= -2ht + 7h \end{aligned}$$

Hence, when $t = 8$, $a = 0.9$,

$$\begin{aligned} -2h(8) + 7h &= -0.9 \\ h &= \mathbf{0.1} \end{aligned}$$

2. (a)

$$a = 4 - 2t$$
$$v = \int 4 - 2t \ dt$$
$$= 4t - t^2 + c$$

When $t = 0$, $v = 5$,

$$\therefore c = 5$$
$$\therefore v = 4t - t^2 + 5$$

At the instantaneous rest, $v = 0$,

$$4t - t^2 + 5 = 0$$
$$t^2 - 4t - 5 = 0$$
$$(t-5)(t+1) = 0$$

$$t = \mathbf{5} \quad \text{or} \quad t = -1 \text{ (N.A.)}$$

(b)

$$s = \int \left(4t - t^2 + 5\right) \ dt$$
$$= 2t^2 - \frac{1}{3}t^3 + 5t + d$$

When $t = 0$, $s = 0$,

$$\therefore d = 0$$
$$\therefore s = 2t^2 - \frac{1}{3}t^3 + 5t$$

When $t = 0$, $s = 0$, when $t = 5$, $s = \frac{100}{3}$, when $t = 6$, $s = 30$

$$\text{Total distance} = 2\left(\frac{100}{3}\right) - 30$$
$$= \mathbf{36\frac{2}{3}\ m}$$

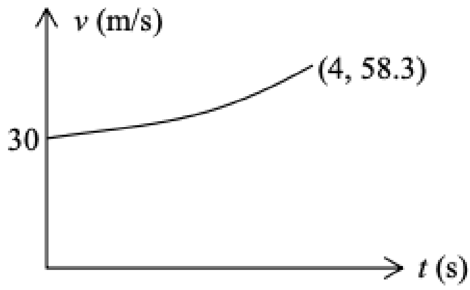
3. (a) At initial velocity, $t = 0$,

$$\begin{aligned} v &= 12e^{k(0)} + 18 \\ &= \mathbf{30\ m/s} \end{aligned}$$

(b) When $t = 2$, $v = 40$

$$\begin{aligned} 40 &= 12e^{2k} + 18 \\ e^{2k} &= \frac{11}{6} \\ k &= \frac{1}{2}\ln\left(\frac{11}{6}\right) \\ &= \mathbf{0.3031\ (3.s.f.)} \end{aligned}$$

(c) Graph



(d)

Area under the curve < Area of trapezium

$$\begin{aligned} \text{Area of trapezium} &= \frac{1}{2}(30 + 60)(4) \\ &= 180 \text{ m} \end{aligned}$$

Hence, the distance travelled will be less than 180 m

□

(e) The maximum acceleration occurs at $t = 4$ where the gradient is the most steep

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= ke^{kt} \end{aligned}$$

$$\begin{aligned} \text{Max acceleration} &= \frac{1}{2}\ln\left(\frac{11}{6}\right) e^{\frac{1}{2}\ln\left(\frac{11}{6}\right)(4)} \\ &= \mathbf{12.2\ m/s^2} \end{aligned}$$

4. (a)

$$a = \frac{t}{2}$$

$$\begin{aligned} v &= \int a \ dt \\ &= \int \frac{t}{2} \ dt \\ &= \frac{1}{4}t^2 + c \end{aligned}$$

When $t = 0$, $v = -1$

$$\begin{aligned} -1 &= \frac{1}{4}(0)^2 + c \\ c &= -1 \end{aligned}$$

$$v = \frac{1}{4}t^2 - 1$$

When $t = 2$,

$$\begin{aligned} v &= \frac{1}{4}(2)^2 - 1 \\ &= \mathbf{0 \ m/s} \end{aligned}$$

(b)

$$\begin{aligned} s &= \int v \ dt \\ &= \int \left(\frac{1}{4}t^2 - 1\right) \ dt \\ &= \frac{1}{12}t^3 - t + d \end{aligned}$$

When $t = 0$, $s = -4$

$$\begin{aligned} -4 &= \frac{1}{2}(0)^3 - (0) + d \\ d &= -4 \end{aligned}$$

$$\therefore s = \frac{1}{12}t^3 - t - 4$$

When $t = 2$,

$$\begin{aligned} s &= \frac{1}{12}(2)^3 - (2) - 4 \\ &= -5\frac{1}{3} \end{aligned}$$

When $t = 5$,

$$\begin{aligned} s &= \frac{1}{12}(5)^3 - 5 - 4 \\ &= 1\frac{5}{12} \end{aligned}$$

Hence,

$$\begin{aligned} \text{Total distance travelled} &= \left(5\frac{1}{3} - 4\right) + \left(1\frac{5}{12} + 5\frac{1}{3}\right) \\ &= \mathbf{8\frac{1}{12} \ m} \end{aligned}$$

5. (a)

$$v = 40e^{-\frac{1}{3}t} - 15$$

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= -\frac{40}{3}e^{-\frac{1}{3}t} \end{aligned}$$

When $t = 0$,

$$\begin{aligned} a &= -\frac{40}{3}e^{-\frac{1}{3}(0)} \\ &= \mathbf{-13\frac{1}{3}\ m/s^2} \end{aligned}$$

(b) When the car stops, $v = 0$,

$$\begin{aligned} 40e^{-\frac{1}{3}t} - 15 &= 0 \\ e^{-\frac{1}{3}t} &= \frac{3}{8} \\ t &= -3\ln\frac{3}{8} \\ &= \mathbf{2.94\ s\ (3.s.f.)} \end{aligned}$$

(c)

$$\begin{aligned} s &= \int v\ dt \\ &= \int \left(40e^{-\frac{1}{3}t} - 15\right)\ dt \\ &= -120e^{-\frac{1}{3}t} - 15t + c \end{aligned}$$

When $t = 0$, $s = 0$

$$c = 120$$

$$\therefore \boldsymbol{s = -120e^{-\frac{1}{3}t} - 15t + 120}$$

(d) To find the braking distance, substitute $t = -3\ln\frac{3}{8}$

$$\begin{aligned} \text{Braking distance} &= -120\left(\frac{3}{8}\right) - 15\left(-3\ln\frac{3}{8}\right) + 120 \\ &= \mathbf{30.9\ m\ (3.s.f.)} \end{aligned}$$