MARCH PRACTICE QUESTIONS 2021
SECONDARY 4 EXPRESS
SECONDARY 5 NORMAL ACADEMIC

ADDITIONAL MATHEMATICS **4047/01**

Specimen Paper Marking Scheme

Date: **3 March 2021** **Duration: NIL**

Candidates answer on separate writing paper

READ THESE INSTRUCTIONS FIRST

Answer all questions.
If working is needed for any question it must be shown with the answer.
Omission of essential working will result in loss of marks.
You are expected to use a scientific calculator to evaluate explicit numerical expressions.
If the degree of accuracy is not specified in the question, and if the answer is not exact, give the answer to three significant figures.
Give answers in degrees to one decimal place.
For π, use either your calculator value of 3.142, unless the question requires the answer in terms of π.

Topic names will be listed above each question for your benefit and revision

Upon completion of solutions:

Each candidate have exactly 2 weeks to submit their solutions
Take a picture or send the digital version of your solutions to me (Kaiwen) via Telegram (**@kaiwen_tutor**) or WhatsApp (**90583779**)
Ensure that all workings are clear and legible
Solutions will be marked based on your presentation, accuracy and completeness of your solutions
A markers' report and the full solutions will be provided at the end of the month

Setter: Ong Kai Wen

This question paper consists of 9 printed pages including the cover page

Content Covered

- Quadratic Functions, Equations & Inequalities
- Indices & Surds
- Polynomials
- Partial Fractions
- Binomial Theorem
- Power, Exponential, Logarithm & Modulus Functions
- Trigonometry
- Linear Law
- Coordinate Geometry
- Further Coordinate Geometry
- Proofs of Plane Geometry
- Calculus

All questions are sourced and selected based on the known abilities of students sitting for the 'O' Level A-Math Examination. If questions are sourced from respective sources, credit will be given when appropriate

Special Note from Tutor (Kaiwen):

Some of these questions are slightly more challenging than others and require some out of the box thinking. When faced with such challenging questions, always go back to the fundamentals and think about the basics you already have learnt in school. Questions will never deviate away from the curriculum that is already pre-set for you

Nonetheless, don't give up if you are unable to solve the questions! Send in your solutions as how you would submit your answer scripts during the National Examinations. From there, I will be able to see and judge the ability of the cohort before moving on and planning the curriculum and content for the rest of the year.

All the best and I really do hope that this initiative will help as many students as it can reach! 加油!

Topic: Quadratic Functions, Equations & Inequalities

The roots of the quadratic equation $\boldsymbol{x^2 - 2(m-1)x + (m^2 - 7) = 0}$ are $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Given that $\boldsymbol{\alpha^2 + \beta^2 = 10}$, find the value of $\boldsymbol{m}$ and obtain an equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$

Solution

$$x^2 - 2(m-1)x + (m^2 - 7) = 0$$

Sum of roots:

$$\alpha + \beta = -\frac{b}{a} = -\frac{\left(-2(m-1)\right)}{1} = 2m - 2$$

Product of roots:

$$\alpha\beta = \frac{c}{a} = \frac{(m^2 - 7)}{1} = m^2 - 7$$

$$\alpha^2 + \beta^2 = 10$$
$$(\alpha + \beta)^2 - 2\alpha\beta = 10$$
$$(2m - 2)^2 - 2(m^2 - 7) = 10$$
$$4m^2 - 8m + 4 - 2m^2 + 14 = 10$$
$$2m^2 - 8m + 8 = 0$$
$$m^2 - 4m + 4 = 0$$
$$(m - 2)^2 = 0$$
$$m = 2$$

∴ **Sum of roots:** $\alpha + \beta = 2(2) - 2 = 2$

∴ **Product of roots:** $\alpha\beta = (2)^2 - 7 = -3$

Since the new roots are $\frac{\alpha}{\beta}$ **and** $\frac{\beta}{\alpha}$,

Sum of new roots:

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = -\frac{10}{3}$$

Product of new roots:

$$\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = \frac{\alpha\beta}{\alpha\beta} = 1$$

∴ **New Equation:**

$$x^2 - \left(-\frac{10}{3}\right)x + 1 = 0$$
$$3x^2 + 10x + 3 = 0$$

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are roots of the equation $\boldsymbol{x^2+x-1=0}$ and $\boldsymbol{a<b}$, prove that

$$\frac{1}{\sqrt{3}}a^2b=\frac{\sqrt{5}+1}{2\sqrt{3}}$$

Solution

$$x^2+x-1=0$$

Given that the roots are a and b,

$$x=\frac{-1\pm\sqrt{(1)^2-4(1)(-1)}}{2(1)}$$

$$=\frac{-1\pm\sqrt{5}}{2}$$

Since $a<b$,

$$\therefore a=\frac{-1-\sqrt{5}}{2}\,,\qquad b=\frac{-1+\sqrt{5}}{2}$$

$$\begin{aligned}
\therefore a^2b&=\left(\frac{-1-\sqrt{5}}{2}\right)^2\cdot\left(\frac{-1+\sqrt{5}}{2}\right)\\
&=\left(\frac{\left(-1-\sqrt{5}\right)\left(-1-\sqrt{5}\right)}{4}\right)\cdot\left(\frac{-1+\sqrt{5}}{2}\right)\\
&=\frac{\left(-1-\sqrt{5}\right)\left(-1-\sqrt{5}\right)\left(-1+\sqrt{5}\right)}{8}\\
&=\frac{\left(-1-\sqrt{5}\right)\left[(-1)^2-\left(\sqrt{5}\right)^2\right]}{8}\\
&=\frac{\left(-1-\sqrt{5}\right)(-4)}{8}\\
&=\frac{4\left(1+\sqrt{5}\right)}{8}\\
&=\frac{1+\sqrt{5}}{2}\\
&=\frac{1+\sqrt{5}}{2}\times\frac{\sqrt{3}}{\sqrt{3}}\\
&=\frac{\left(\sqrt{5}+1\right)\left(\sqrt{3}\right)}{2\sqrt{3}}
\end{aligned}$$

Hence:

$$\therefore \frac{1}{\sqrt{3}}a^2b=\frac{\sqrt{5}+1}{2\sqrt{3}}\ (\textbf{shown})$$

Show that $\boldsymbol{my = x^2 - 4(x-1)}$ meets the curve $\boldsymbol{y = x^2 - 3x + 2}$ at 2 distinct points for all non-zero values of $\boldsymbol{m}$

Solution

$$\boldsymbol{my = x^2 - 4(x-1) \dots\dots\dots\dots\dots (1)}$$
$$\boldsymbol{y = x^2 - 3x + 2 \dots\dots\dots\dots\dots (2)}$$

From Equation (1),

$$\boldsymbol{y = \frac{x^2 - 4(x-1)}{m} \dots\dots\dots\dots\dots (3)}$$

To show that there are 2 distinct intersection points, we assume that the 2 curves meet and show that it holds for all non-zero values of m.

$$\boldsymbol{(3) = (2)}$$

$$\boldsymbol{\frac{x^2 - 4(x-1)}{m} = x^2 - 3x + 2}$$
$$\boldsymbol{x^2 - 4(x-1) = m\left(x^2 - 3x + 2\right)}$$
$$\boldsymbol{x^2 - 4(x-1) = mx^2 - 3mx + 2m}$$
$$\boldsymbol{x^2 - mx^2 - 4x + 3mx + 4 - 2m = 0}$$
$$\boldsymbol{(1-m)x^2 + (3m-4)x + (4-2m) = 0}$$

Since we assume that the 2 curves meet at 2 distinct points, $b^2 - 4ac > 0$

$$\boldsymbol{\therefore (3m-4)^2 - 4(1-m)(4-2m) > 0}$$
$$\boldsymbol{9m^2 - 24m + 16 - 4[4 - 6m + 2m^2] > 0}$$
$$\boldsymbol{9m^2 - 24m + 16 - 16 - 24m - 8m^2 > 0}$$
$$\boldsymbol{m^2 > 0}$$
$$\boldsymbol{m < 0 \quad \text{or} \quad m > 0}$$

With the given equality, m can take on both positive and negative values. m can hold for all non-zero values of m. Hence, the 2 curve meets at 2 distinct points. (shown)

Solve the simultaneous equations

$$9^y \times 27 = 3^{2x-1}$$
$$7^y\sqrt{7^x} = 343$$

Solution

$9^y \times 27 = 3^{2x-1} \dots\dots\dots\dots\dots (1)$

$7^y\sqrt{7^x} = 343 \dots\dots\dots\dots\dots (2)$

From Equation (1)

$9^y \times 27 = 3^{2x-1}$

$3^{2y} \times 3^3 = 3^{2x-1}$

$3^{2y+3} = 3^{2x-1}$

$\therefore 2y + 3 = 2x - 1$

$y = x - 2 \dots\dots\dots\dots\dots (3)$

From Equation (2)

$7^y\sqrt{7^x} = 343$

$(7^y)\left(7^{\frac{x}{2}}\right) = 7^3$

$7^{y+\frac{x}{2}} = 7^3$

$\therefore y + \frac{x}{2} = 3$

$y = 3 - \frac{x}{2} \dots\dots\dots\dots\dots (4)$

$(3) = (4)$

$x - 2 = 3 - \frac{x}{2}$

$2x - 4 = 6 - x$

$3x = 10$

$x = \frac{10}{3}$

$\quad = 3\frac{1}{3}$

Hence, we substitute $x = 3\frac{1}{3}$ into Equation (3)

$y = \frac{10}{3} - 2$

$\quad = \frac{4}{3}$

$\quad = 1\frac{1}{3}$

Topic: Indices & Surds

Without using a calculator, find the value of $\boldsymbol{a}$ and $\boldsymbol{b}$ for which

$$\frac{6}{\sqrt{2}}\left(\frac{5\sqrt{32}}{2}+\frac{15}{\sqrt{50}}-\frac{14}{7\sqrt{6}}\right)=a-b\sqrt{3}$$

Solution

$$\begin{aligned}
\text{LHS} &= \frac{6}{\sqrt{2}}\left(\frac{5\sqrt{32}}{2}+\frac{15}{\sqrt{50}}-\frac{14}{7\sqrt{6}}\right)\\
&= \left(\frac{6}{\sqrt{2}}\right)\left(\frac{5\sqrt{32}}{2}\right)+\left(\frac{6}{\sqrt{2}}\right)\left(\frac{15}{\sqrt{50}}\right)-\left(\frac{6}{\sqrt{2}}\right)\left(\frac{14}{7\sqrt{6}}\right)\\
&= \frac{30(\sqrt{2^4})(\sqrt{2})}{2\sqrt{2}}+\frac{90}{(\sqrt{2})(\sqrt{2})(\sqrt{5^2})}-\frac{84}{7(\sqrt{2})(\sqrt{2})(\sqrt{3})}\\
&= \frac{30(2^2)}{2}+\frac{90}{2(5)}-\frac{84}{7(2)(\sqrt{3})}\\
&= 15(4)+9-\frac{6}{\sqrt{3}}\\
&= 69-\frac{6}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\
&= 69-\frac{6\sqrt{3}}{3}\\
&= 69-2\sqrt{3}
\end{aligned}$$

$\therefore a=69, b=2$

Topic: Polynomials

Given that $4x^4 - px^3 - 9x^2 + x + 2 = (x-2)(x+1)(ax^2 - b)$, find a, b and p

Solution

$$4x^4 - px^3 - 9x^2 + x + 2 = (x-2)(x+1)(ax^2 - b)$$

$$\begin{aligned}\text{RHS} &= (x-2)(x+1)(ax^2-b) \\ &= (x^2 - x - 2)(ax^2 - b) \\ &= ax^4 - bx^2 - ax^3 + bx - 2ax^2 + 2b \\ &= ax^4 - ax^3 - (2a+b)x^2 + bx + 2b\end{aligned}$$

Comparing coefficients,

$a = 4$

$2b = 2 \Rightarrow b = 1$

$-a = -p \Rightarrow p = 4$

Topic: Partial Fractions

Express the following in partial fractions:

$$\frac{123x - 46}{(2 - x)(3x - 1)^2}$$

Solution

$$\frac{123x - 46}{(2 - x)(3x - 1)^2} = \frac{A}{2 - x} + \frac{B}{3x - 1} + \frac{C}{(3x - 1)^2}$$

$$\begin{aligned} 123x - 46 &= A(3x - 1)^2 + B(2 - x)(3x - 1) + C(2 - x) \\ &= A\left(9x^2 - 6x + 1\right) + B\left(6x - 2 - 3x^2 + x\right) + 2C - Cx \\ &= 9Ax^2 - 6Ax + A + 7Bx - 2B - 3Bx^2 + 2C - Cx \\ &= (9A - 3B)x^2 + (7B - 6A - C)x + (A - 2B + 2C) \end{aligned}$$

Comparing coefficients,

$9A - 3B = 0 \ldots\ldots\ldots\ldots\ldots (1)$

$7B - 6A - C = 123 \ldots\ldots\ldots\ldots\ldots (2)$

$A - 2B + 2C = -46 \ldots\ldots\ldots\ldots\ldots (3)$

From Equation (1),

$3B = 9A \Rightarrow B = 3A \ldots\ldots\ldots\ldots\ldots (4)$

Substitute Equation (4) into Equation (2) and Equation (3),

$7(3A) - 6A - C = 123$

$C = 15A - 123 \ldots\ldots\ldots\ldots\ldots (5)$

$A - 2(3A) + 2C = -46$

$2C - 5A = -46 \ldots\ldots\ldots\ldots\ldots\ldots (6)$

Substitute Equation (5) into Equation (6),

$2(15A - 123) - 5A = -46$

$30A - 246 - 5A = -46$

$25A = 200 \Rightarrow A = 8$

Substitute $A = 8$ into Equation (4) and Equation (5)

$B = 3(8) \Rightarrow B = 24$

$C = 15(8) - 123 \Rightarrow C = -3$

$$\therefore \frac{123x - 46}{(2 - x)(3x - 1)^2} = \frac{8}{2 - x} + \frac{24}{3x - 1} - \frac{3}{(3x - 1)^2}$$

Topic: Binomial Theorem

In the given expansion, where $n > 0$, the first 3 terms in ascending powers of x are

$$\left(\frac{k}{x^3} + cx\right)^n = \frac{1}{x^{66}} + \frac{44}{x^{62}} + \frac{924}{x^{58}} + \cdots$$

(a) Find the values of k, n and c

(b) In the given expansion, where p is a positive constant, the term independent of x is 5376. Find the value of p

$$\left(x^2 - \frac{p}{2x}\right)^9$$

[S4 CCHY P2/2012 PRELIM Qn 2]

Solution

(a) To find the values, we expand the left hand side of the equation first

$$\left(\frac{k}{x^3} + cx\right)^n = \left(\frac{k}{x^3}\right)^n + \binom{n}{1}\left(\frac{k}{x^3}\right)^{n-1}(cx)^1 + \binom{n}{2}\left(\frac{k}{x^3}\right)^{n-2}(cx)^2 + \cdots$$

$$= \frac{k^n}{x^{3n}} + n\left(\frac{k^{n-1}}{x^{3n-3}}\right)(cx) + \frac{n(n-1)}{2}\left(\frac{k^{n-2}}{x^{3n-6}}\right)(c^2x^2) + \cdots$$

$$= \frac{k^n}{x^{3n}} + \frac{nck^{n-1}}{x^{3n-4}} + \frac{n(n-1)c^2k^{n-2}}{2x^{3n-6}} + \cdots$$

Comparing coefficients,

$x^{3n} = x^{66}$

$3n = 66$

$n = 22$

$k^{22} = 1$

$k = 1$

$nck^{n-1} = 44$

$(22)(c)(1) = 44$

$c = 2$

(b) To find the value of p, we need to find the general $(r+1)$th term

$$\textbf{General } (r+1)\textbf{th term} = \binom{9}{r}\left(x^2\right)^{9-r}\left(-\frac{p}{2x}\right)^r$$

$$= \binom{9}{r}\left(x^{18-2r}\right)\left(-\frac{p}{2}\right)^r\left(x^{-r}\right)$$

$$= \binom{9}{r}\left(x^{18-3r}\right)\left(-\frac{p}{2}\right)^r$$

For the independent term of x, x^0

$x^0 = x^{18-3r}$

$18 - 3r = 0$

$r = 6$

Given that the independent term of x is 5376

$\therefore \binom{9}{6}\left(-\frac{p}{2}\right)^6 = 5376$

$84\left(\frac{p^6}{64}\right) = 5376$

$p^6 = 4096$

$p = 4$

Topic: Power, Exponential, Logarithm & Modulus Functions
Solve the following equation

$$\log_9 3x - \log_3 3x = \log_{\sqrt{3}} 3$$

Solution

$$\log_9 3x - \log_3 3x = \log_{\sqrt{3}} 3$$

$$\frac{\log_3 3x}{\log_3 9} - \log_3 3x = \frac{\log_3 3}{\log_3 \sqrt{3}}$$

$$\frac{\log_3 3x}{\log_3 (3)^2} - \log_3 3x = \frac{\log_3 3}{\log_3 (3)^{\frac{1}{2}}}$$

$$\frac{\log_3 3x}{2\log_3 3} - \log_3 3x = \frac{\log_3 3}{\frac{1}{2}\log_3 3}$$

$$\frac{\log_3 3x}{2} - \log_3 3x = \frac{1}{\left(\frac{1}{2}\right)}$$

$$\frac{1}{2}\log_3 3x - \log_3 3x = 2$$

$$-\frac{1}{2}\log_3 3x = 2$$

$$\log_3 3x = -4$$

$$\therefore 3x = (3)^{-4}$$

$$3x = \frac{1}{81}$$

$$x = \frac{1}{243}$$

Solve the equation

$$\log_4(3x-1)^2 - \frac{2}{\log_{\sqrt{2}} 2} = \log_4\left(x^2 + \frac{9}{4}\right)$$

Solution

$$\log_4(3x-1)^2 - \frac{2}{\log_{\sqrt{2}} 2} = \log_4\left(x^2 + \frac{9}{4}\right)$$

$$\log_4(3x-1)^2 - \frac{2}{\left(\frac{1}{\log_2 \sqrt{2}}\right)} = \log_4\left(x^2 + \frac{9}{4}\right)$$

$$\log_4(3x-1)^2 - 2\log_2 \sqrt{2} = \log_4\left(x^2 + \frac{9}{4}\right)$$

$$\log_4(3x-1)^2 - 2\log_2 2^{\frac{1}{2}} = \log_4\left(x^2 + \frac{9}{4}\right)$$

$$\log_4(3x-1)^2 - 1 = \log_4\left(x^2 + \frac{9}{4}\right)$$

$$\log_4(3x-1)^2 - \log_4 4 = \log_4\left(x^2 + \frac{9}{4}\right)$$

$$\log_4\left(\frac{(3x-1)^2}{4}\right) = \log_4\left(x^2 + \frac{9}{4}\right)$$

$$\therefore \frac{(3x-1)^2}{4} = x^2 + \frac{9}{4}$$

$$9x^2 - 6x + 1 = 4x^2 + 9$$

$$5x^2 - 6x - 8 = 0$$

$$(x-2)(5x+4) = 0$$

$$x = 2 \quad \text{or} \quad x = -\frac{4}{5}$$

Topic: Trigonometry

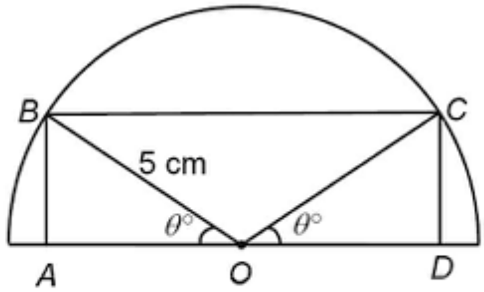
Prove that

$$\frac{\sin A + \sin 2A + \sin 3A}{\cos A + \cos 2A + \cos 3A} = \tan 2A$$

Solution

$$\frac{\sin A + \sin 2A + \sin 3A}{\cos A + \cos 2A + \cos 3A} = \tan 2A$$

$$\begin{aligned}
\text{LHS} &= \frac{\sin A + \sin 2A + \sin 3A}{\cos A + \cos 2A + \cos 3A} \\
&= \frac{\sin(2A - A) + \sin 2A + \sin(2A + A)}{\cos(2A - A) + \cos 2A + \cos(2A + A)} \\
&= \frac{\sin 2A \cos A - \cos 2A \sin A + \sin 2A + \sin 2A \cos A + \cos 2A \sin A}{\cos 2A \cos A + \sin 2A \sin A + \cos 2A + \cos 2A \cos A - \sin 2A \sin A} \\
&= \frac{2 \sin 2A \cos A + \sin 2A}{2 \cos 2A \cos A + \cos 2A} \\
&= \frac{\sin 2A\,(2 \cos A + 1)}{\cos 2A\,(2 \cos A + 1)} \\
&= \frac{\sin 2A}{\cos 2A} \\
&= \tan 2A \\
&= \text{RHS (shown)}
\end{aligned}$$



The diagram on the right shows a rectangle $\boldsymbol{ABCD}$ inside a semicircle, centre $\boldsymbol{O}$ and radius $\mathbf{5}\boldsymbol{cm}$, such that $\angle \boldsymbol{BOA} = \angle \boldsymbol{COD} = \boldsymbol{\theta}°$

(a) Show that the perimeter, $\boldsymbol{P}\ \boldsymbol{cm}$, of the rectangle is given by the formula $\boldsymbol{P} = \mathbf{20}\cos\boldsymbol{\theta} + \mathbf{10}\sin\boldsymbol{\theta}$

(b) Express $\boldsymbol{P}$ in the form $\boldsymbol{R}\cos(\boldsymbol{\theta} - \boldsymbol{\alpha})$ and hence find the value of $\boldsymbol{\theta}$ for which $\boldsymbol{P} = \mathbf{16}$

(c) Find the value of $\boldsymbol{k}$ for which the area of the rectangle is $\boldsymbol{k}\sin 2\boldsymbol{\theta}\ \boldsymbol{cm}^2$

Solution

(a) **In Triangle $\boldsymbol{AOB}$,**

$$\sin\theta = \frac{AB}{5} \Rightarrow AB = 5\sin\theta$$

$$\cos\theta = \frac{AO}{5} \Rightarrow AO = 5\cos\theta$$

Perimeter of rectangle $\boldsymbol{ABCD} = \boldsymbol{AB} + \boldsymbol{BC} + \boldsymbol{CD} + \boldsymbol{AD}$

$$= 2AB + 4AO\ (\because AB = CD, BC = AD = 2AO)$$

$$= 2(5\sin\theta) + 4(5\cos\theta)$$

$$= 20\cos\theta + 10\sin\theta\ \ (\text{shown})$$

(b) $P = 20\cos\theta + 10\sin\theta$

$$R = \sqrt{(20)^2 + (10)^2}$$
$$= \sqrt{500}$$
$$= 10\sqrt{5}$$

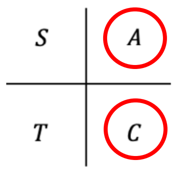
$$\tan\alpha = \frac{10}{20}$$
$$\alpha = \tan^{-1}\left(\frac{10}{20}\right)$$
$$= 26.56505\ldots$$
$$= 26.6^\circ\ (3.\text{s.f.})$$

$$\therefore P = 10\sqrt{5}\cos(\theta - 26.6^\circ)$$

Since $P = 16$,

$$\therefore 16 = 10\sqrt{5}\cos(\theta - 26.6^\circ)$$
$$\cos(\theta - 26.6^\circ) = \frac{16}{10\sqrt{5}}$$

$$\text{Basic Angle } \alpha = \cos^{-1}\left(\frac{16}{10\sqrt{5}}\right)$$



$$\theta - 26.6^\circ = \cos^{-1}\left(\frac{16}{10\sqrt{5}}\right) \quad \text{or} \quad \theta - 26.6^\circ = 360^\circ - \cos^{-1}\left(\frac{16}{10\sqrt{5}}\right)\ \text{(rejected)}$$

$$\therefore \theta = \cos^{-1}\left(\frac{16}{10\sqrt{5}}\right) + \tan^{-1}\left(\frac{10}{20}\right)$$
$$= 70.877435\ldots$$
$$= 70.9^\circ$$

(c) $\text{Area of rectangle} = AB \times BC$

$$= AB \times 2AO\ (\because BC = 2AO)$$
$$= 5\sin\theta \times 2[5\cos\theta]$$
$$= 50\sin\theta\cos\theta$$
$$= 25\sin 2\theta$$

$$\therefore k = 25$$

Prove that, for all real values of x

$$\frac{\cot x}{\sqrt{1+\cot^2 x}} - \frac{\operatorname{cosec} x}{\tan x + \cot x} = 0$$

Solution

$$\frac{\cot x}{\sqrt{1+\cot^2 x}} - \frac{\operatorname{cosec} x}{\tan x + \cot x} = 0$$

$$\begin{aligned}
\text{LHS} &= \frac{\cot x}{\sqrt{1+\cot^2 x}} - \frac{\operatorname{cosec} x}{\tan x + \cot x} \\
&= \frac{\cot x}{\sqrt{\operatorname{cosec}^2 x}} - \frac{\operatorname{cosec} x}{\tan x + \cot x} \\
&= \frac{\cot x}{\operatorname{cosec} x} - \frac{\operatorname{cosec} x}{\left(\frac{\sin x}{\cos x}\right) + \left(\frac{\cos x}{\sin x}\right)} \\
&= \frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)} - \frac{\left(\frac{1}{\sin x}\right)}{\left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right)} \\
&= \left(\frac{\cos x}{\sin x} \div \frac{1}{\sin x}\right) - \left(\frac{1}{\sin x} \div \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right) \\
&= \left(\frac{\cos x}{\sin x} \times \sin x\right) - \left(\frac{1}{\sin x} \times \sin x \cos x\right) \\
&= \cos x - \cos x \\
&= 0 \\
&= \text{RHS (shown)}
\end{aligned}$$

Solve for y, between $0°$ and $360°$ for the equation

$$4\operatorname{cosec}^2 y = 7 - \cot^2 y + 2\cot y$$

<u>**Solution**</u>

$$4\operatorname{cosec}^2 y = 7 - \cot^2 y + 2\cot y$$
$$4(1 + \cot^2 y) = 7 - \cot^2 y + 2\cot y$$
$$4 + 4\cot^2 y = 7 - \cot^2 y + 2\cot y$$
$$5\cot^2 y - 2\cot y - 3 = 0$$

Let $x = \cot y$,

$$5x^2 - 2x - 3 = 0$$
$$(x-1)(5x+3) = 0$$

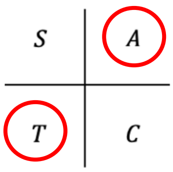
$x = 1$ or $x = -\dfrac{3}{5}$

$\cot y = 1$ or $\cot y = -\dfrac{3}{5}$

$\tan y = 1$ or $\tan y = -\dfrac{5}{3}$

$\tan y = 1$

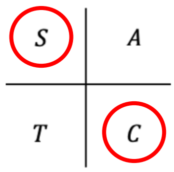
Basic Angle $\alpha = 45°$



$y = 45°$ or $y = 180° + 45°$
$= 225°$

$\tan y = -\dfrac{5}{3}$

Basic Angle $\alpha = \tan^{-1}\left(\dfrac{5}{3}\right)$



$y = 180° - \tan^{-1}\left(\dfrac{5}{3}\right)$ or $y = 360° - \tan^{-1}\left(\dfrac{5}{3}\right)$

$= 120.963756\ldots$ $\quad$ $= 300.963756\ldots$

$= 121.0°(1.\text{d.p.})$ $\quad$ $= 301.0°(1.\text{d.p.})$

Topic: Linear Law

Variables x and y are related by the equation $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$, where a and b are constants. The table below shows measured values of x and y

x	1	2	3	4	5	6
y	5.7	5.6	5.9	6.2	6.6	6.9

(a) On a piece of graph paper, plot $y\sqrt{x}$ against x, using a scale of $2\ cm$ to represent 1 unit on the $y\sqrt{x}$ axis. Draw a straight line graph to represent the equation $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$

(b) Use your graph to estimate the value of a and of b

(c) On the same diagram, draw the line representing the equation $y = \frac{3x}{\sqrt{x}}$ and hence find the value for which

$$x = \frac{b}{3 - a}$$

Solution

(a) **To show the axes,**

$$y = a\sqrt{x} + \frac{b}{\sqrt{x}}$$

$$y\sqrt{x} = ax + b \ldots\ldots\ldots (1)$$

x	1	2	3	4	5	6
y	5.7	5.6	5.9	6.2	6.6	6.9
$y\sqrt{x}$	5.7	7.9	10.2	12.4	14.8	16.9

$Y = mX + C$ **where a is the gradient, and b is the y-intercept**

Graph is drawn on the next page

(b) **To find a, we need to find the gradient of the line**

$$a = \frac{14.8 - 5.7}{5 - 1}$$

$$= 2.275 \ldots$$

$$= 2.28\ (3.\mathrm{s.f.})$$

To find b, we can read the y-intercept off the graph

$$b = 3.3$$

(c) To find the line to sketch

$$\boldsymbol{y = \frac{3x}{\sqrt{x}}}$$

$$\boldsymbol{y\sqrt{x} = 3x \dots\dots\dots (2)}$$

Graph is drawn on the next page

To find the value of the following,

$$\boldsymbol{x = \frac{b}{3-a}}$$

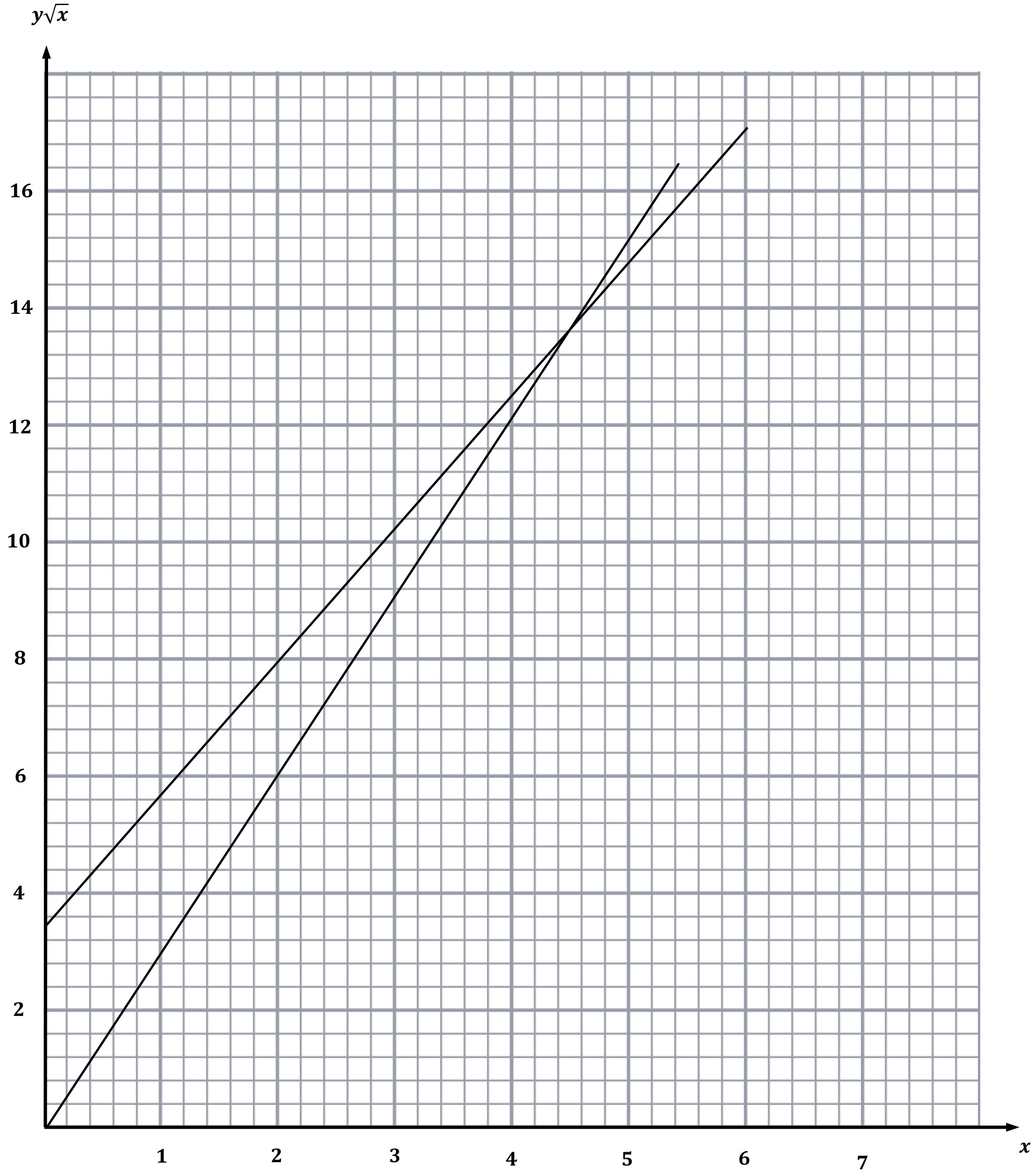
$$\boldsymbol{3x - ax = b}$$

$$\boldsymbol{3x = ax + b}$$

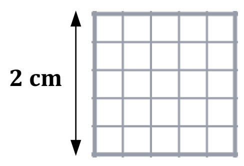
We can obtain $3x = ax + b$ by equating Equation (1) and Equation (2). Hence, to find the value of the following, we are looking for the intersection point between the 2 lines.

$\therefore$ From the graph, $x = 4.5$

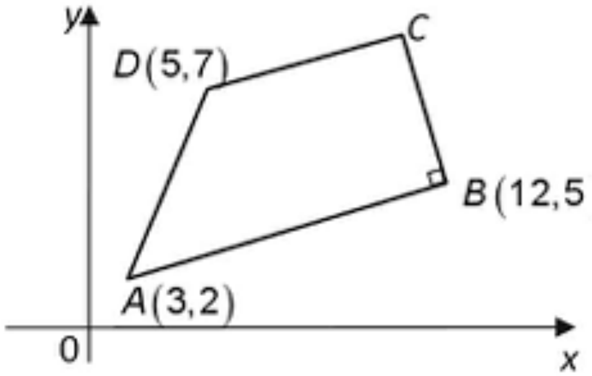
Graphical Solution for Linear Law Question



Legend



Topic: Coordinate Geometry



Solutions to this question by accurate drawing will not be accepted

In the diagram on the right, AB is parallel to DC and $\angle ABC = 90°$. Given that the coordinates of A, B and D are $(3, 2)$, $(12, 5)$ and $(5, 7)$ respectively, find

(a) The equations of BC and DC

(b) The coordinates of C

(c) The equation of the perpendicular bisector of AB

Solution

(a) **Since AB is parallel to DC, the gradients for both AB and DC are the same**

$$\textbf{Gradient of } AB = \textbf{Gradient of } DC = \frac{5-2}{12-3}$$
$$= \frac{1}{3}$$

$$\therefore y - 7 = \frac{1}{3}(x-5)$$
$$3y - 21 = x - 5$$
$$3y = x + 16 \text{ (Equation of } DC\text{)}$$

Since AB is perpendicular to BC, gradient of $AB \times$ gradient of $BC = -1$

$$\therefore \textbf{Gradient of } BC = \frac{-1}{\left(\frac{1}{3}\right)}$$
$$= -3$$

$$\therefore y - 5 = -3(x - 12)$$
$$y = -3x + 41 \text{ (Equation of } BC\text{)}$$

(b) **To find coordinate of C, we find the intersection of BC and CD**

$\mathbf{3y = x + 16 \ldots\ldots\ldots\ldots\ldots (1)}$

$\mathbf{y = -3x + 41 \ldots\ldots\ldots\ldots\ldots (2)}$

Substitute Equation (2) into Equation (1),

$\mathbf{3(-3x + 41) = x + 16}$

$\mathbf{-9x + 123 = x + 16}$

$\mathbf{10x = 107}$

$\mathbf{x = 10.7}$

Substitute $x = 10.7$ into Equation (2)

$\therefore \mathbf{y = -3(10.7) + 41}$

$\mathbf{\quad = 8.9}$

$\therefore$ **Coordinates of $C = (10.7, 8.9)$**

(c) **At the perpendicular bisector, it cuts AB at the midpoint**

$$\therefore \text{Midpoint of } AB = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{3 + 12}{2}, \frac{2 + 5}{2}\right)$$

$$= \left(\frac{15}{2}, \frac{7}{2}\right)$$

Gradient of the perpendicular bisector is the same as the gradient of BC

$\therefore$ **Gradient of the perpendicular bisector $= -3$**

$$\therefore y - \frac{7}{2} = -3\left(x - \frac{15}{2}\right)$$

$\mathbf{2y - 7 = -6x + 45}$

$\mathbf{2y = -6x + 52}$

$\mathbf{y = -3x + 26}$

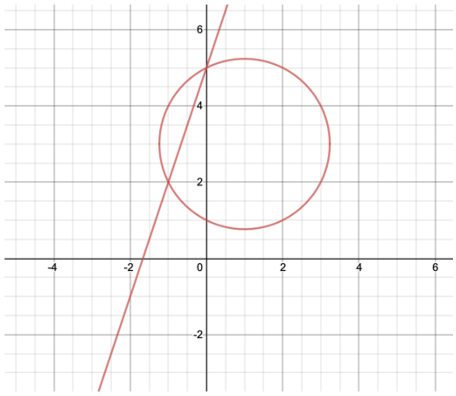
Topic: Further Coordinate Geometry

The straight line $3x - y + 5 = 0$ and the curve $x^2 + y^2 - 2x - 6y + 5 = 0$ intersect 2 points A and B. Find the coordinates of A and B. Hence, find the length of AB

Solution

$3x - y + 5 = 0 \ldots\ldots\ldots\ldots\ldots (1)$

$x^2 + y^2 - 2x - 6y + 5 = 0 \ldots\ldots\ldots\ldots\ldots (2)$



From Equation (1),

$y = 3x + 5 \ldots\ldots\ldots\ldots\ldots (3)$

Substitute Equation (3) into Equation (2),

$x^2 + (3x + 5)^2 - 2x - 6(3x + 5) + 5 = 0$

$x^2 + 9x^2 + 30x + 25 - 2x - 18x - 30 + 5 = 0$

$10x^2 + 10x = 0$

$x^2 + x = 0$

$x(x + 1) = 0$

$x = 0 \quad \text{or} \quad x = -1$

Substitute $x = 0$ into Equation (3),

$y = 3(0) + 5$

$= 5$

$\therefore$ Coordinate $A = (0, 5)$

Substitute $x = -1$ into Equation (3),

$y = 3(-1) + 5$

$= 2$

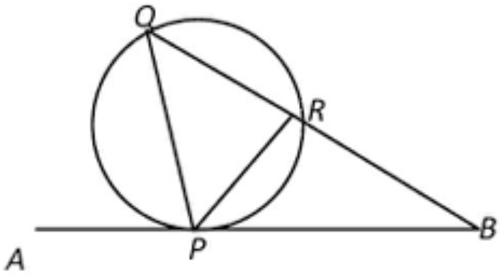
$\therefore$ Coordinate $B = (-1, 2)$

$\therefore$ Length of $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{\big((0) - (-1)\big)^2 + \big((5) - (2)\big)^2}$

$= \sqrt{10}$ units

Topic: Proofs of Plane Geometry



The diagram on the right shows a triangle $\boldsymbol{PQR}$ inscribed in a circle and $\boldsymbol{AB}$ is a tangent to the circle at $\boldsymbol{P}$. $\boldsymbol{PR}$ bisects $\angle \boldsymbol{QPB}$. Prove that

(a) $\boldsymbol{RQ = RP}$

(b) $\boldsymbol{BR \times RP = BP^2 - BR^2}$

Solution

(a) **Let** $\boldsymbol{\measuredangle BPR = \theta°}$

$\boldsymbol{\measuredangle BPR = \measuredangle PQR}$ **[$\measuredangle$ in alternate segment]**

Since $\boldsymbol{PR}$ **bisects** $\boldsymbol{\measuredangle QPB}$**, it splits** $\boldsymbol{\measuredangle QPB}$ **into 2 equal parts [Definition of $\measuredangle$ bisector]**

$\boldsymbol{\measuredangle BPR = \measuredangle RPQ = \theta°}$

Hence, in $\boldsymbol{\Delta PQR, \measuredangle RPQ = \measuredangle PQR = \theta°}$ **[$\measuredangle$ in an isosceles triangle]**

This implies that $\boldsymbol{\Delta PQR}$ **is an isosceles triangle**

$\boldsymbol{\therefore RQ = RP}$ **(shown)**

(b) $\boldsymbol{\measuredangle BRP = 2\theta°}$ **[exterior $\measuredangle$ = sum of interior opposite $\measuredangle$]**

$\boldsymbol{\measuredangle BPQ = 2\theta°}$ **[$\boldsymbol{\measuredangle BPR = \measuredangle RPQ}$]**

$\boldsymbol{\measuredangle BPR = \measuredangle PQR = \theta°}$ **[$\measuredangle$ in alternate segment]**

$\boldsymbol{\therefore \Delta BPR}$ **is similar to** $\boldsymbol{\Delta BQP}$ **[AA similarity test]**

By similarity ratios, $\boldsymbol{\frac{BP}{BR} = \frac{PQ}{RP} = \frac{QB}{PB}}$

$\boldsymbol{\frac{BP}{BR} = \frac{QB}{PB}}$

$\boldsymbol{\Rightarrow BP^2 = QB \times BR}$

$\boldsymbol{QB = BR + RQ}$ **(given)**

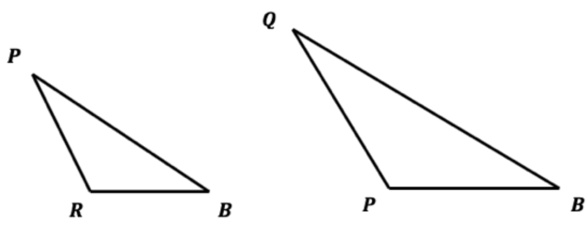
$\boldsymbol{\therefore BP^2 = (BR + RQ) \times BR}$

$\boldsymbol{BP^2 = BR^2 + RQ \times BR}$

$\boldsymbol{BR \times RQ = BP^2 - BR^2}$

Since $\boldsymbol{RQ = RP}$ **[$\boldsymbol{\Delta PQR}$ is isosceles]**

$\boldsymbol{\therefore BR \times RP = BP^2 - BR^2}$ **(shown)**



Topic: Calculus

Show that

$$\frac{d}{dx}\left(\frac{x}{1+5x}\right)=\frac{1}{(1+5x)^2}$$

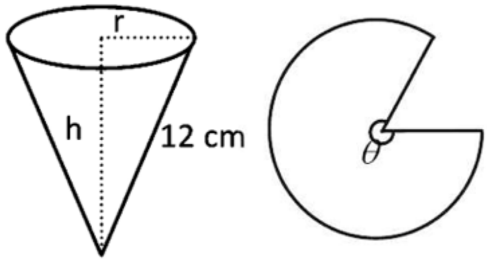
Hence, or otherwise, find the area bounded by the x-axis, the lines $x=1$, the line $x=3$, and the curve

$$y=\left(\frac{4}{1+5x}\right)^2$$

Solution

$$\begin{aligned}\frac{d}{dx}\left(\frac{x}{1+5x}\right)&=\frac{(1+5x)(1)-(x)(5)}{(1+5x)^2}\\&=\frac{1+5x-5x}{(1+5x)^2}\\&=\frac{1}{(1+5x)^2}\ \text{(shown)}\end{aligned}$$

$$\begin{aligned}\therefore\int_1^3\left(\frac{4}{1+5x}\right)^2dx&=\int_1^3\frac{16}{(1+5x)^2}\,dx\\&=16\int_1^3\frac{1}{(1+5x)^2}\,dx\\&=16\left[\frac{x}{1+5x}\right]_1^3\\&=16\left[\frac{(3)}{1+5(3)}-\frac{(1)}{1+5(1)}\right]\\&=\frac{1}{3}\end{aligned}$$



Peter wishes to make a cone to hold water. He makes a right circular cone, without overlap, of depth $\boldsymbol{h}$ **cm**, radius $\boldsymbol{r}$ **cm** and slant height **12 cm**

(a) Show that the volume of the cone, $\boldsymbol{V}$ $\mathbf{cm^3}$ is given by the equation

$$V = 48\pi h - \frac{1}{3}\pi h^3$$

(b) Find the value of $\boldsymbol{h}$ for which $\boldsymbol{V}$ has a stationary value. Find this value of $\boldsymbol{V}$ and determine whether it is a maximum or minimum

(c) Given that Peter uses material that is originally in the form of a sector of angle $\boldsymbol{\theta}$ radians. Show that $\boldsymbol{\theta}$ is approximately **5.13** when $\boldsymbol{V}$ is stationary

<u>**Solution**</u>

(a) **By Pythagoras' Theorem,**

$r^2 + h^2 = 12^2$

$r^2 = 144 - h^2$

$$\therefore \text{Volume of cone} = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi h(144 - h^2)$$
$$= 48\pi h - \frac{1}{3}\pi h^3 \text{ (shown)}$$

(b) **Since we know that V has a stationary value, $V' = 0$**

$V = 48\pi h - \frac{1}{3}\pi h^3$

$V' = 48\pi - \pi h^2$

$\therefore 48\pi - \pi h^2 = 0$

$\pi h^2 = 48\pi$

$h^2 = 48$

$h = \sqrt{48}\ (\text{rej} - \sqrt{48})$

$= 6.928203 \ldots$

$= 6.93 \text{ cm } (3.\text{s.f.})$

$\therefore V = 48\pi(\sqrt{48}) - \frac{1}{3}\pi(\sqrt{48})^3$

$= 696.4989559 \ldots$

$= 696 \text{ cm}^3\ (3.\text{s.f})$

To determine its nature, we perform the second derivative test

$$V' = 48\pi - \pi h^2$$

$$V'' = -2\pi h$$

$$= -2\pi(\sqrt{48})$$

$$= -43.531184 \ldots < 0$$

Since $V'' < 0$, this is a maximum value

(c) Surface area $= \pi r l$

$$= \pi\left(\sqrt{144 - h^2}\right)(12)$$

$$= \pi\left(\sqrt{144 - (\sqrt{48})^2}\right)(12)$$

$$= 12\sqrt{96}\pi$$

$$\therefore \text{Area of sector} = \frac{1}{2}r^2\theta$$

$$12\sqrt{96}\pi = \frac{1}{2}l^2\theta$$

$$12\sqrt{96}\pi = \frac{1}{2}(12)^2\theta$$

$$\theta = 5.13019932 \ldots$$

$$= 5.13 \text{ rad } (3.\text{s.f.}) \text{ (shown)}$$

Evaluate the following

$$\int \frac{2x^2+16x}{(1-3x)(2x+1)^2}\,dx$$

Solution

To solve this integral, we first need to perform partial fractions

$$\frac{2x^2+16x}{(1-3x)(2x+1)^2} = \frac{A}{1-3x} + \frac{B}{2x+1} + \frac{C}{(2x+1)^2}$$

$$2x^2+16x = A(2x+1)^2 + B(1-3x)(2x+1) + C(1-3x)$$

Let $x = -\frac{1}{2}$,

$$2\left(-\frac{1}{2}\right)^2 + 16\left(-\frac{1}{2}\right) = A\left(2\left(-\frac{1}{2}\right)+1\right)^2 + B\left(1-3\left(-\frac{1}{2}\right)\right)\left(2\left(-\frac{1}{2}\right)+1\right) + C\left(1-3\left(-\frac{1}{2}\right)\right)$$

$$\frac{5}{2}C = -\frac{15}{2}$$

$$C = -3$$

Let $x = \frac{1}{3}$,

$$2\left(\frac{1}{3}\right)^2 + 16\left(\frac{1}{3}\right) = A\left(2\left(\frac{1}{3}\right)+1\right)^2 + B\left(1-3\left(\frac{1}{3}\right)\right)\left(2\left(\frac{1}{3}\right)+1\right) + C\left(1-3\left(\frac{1}{3}\right)\right)$$

$$\frac{25}{9}A = \frac{50}{9}$$

$$A = 2$$

Let $x = 1$,

$$2(1)^2 + 16(1) = A(2(1)+1)^2 + B\big(1-3(1)\big)(2(1)+1) + C\big(1-3(1)\big)$$

$$6B = 6$$

$$B = 1$$

$$\therefore \frac{2x^2+16x}{(1-3x)(2x+1)^2} = \frac{2}{1-3x} + \frac{1}{2x+1} - \frac{3}{(2x+1)^2}$$

$$\begin{aligned}
\therefore \int \frac{2x^2+16x}{(1-3x)(2x+1)^2}\,dx &= \int \left[\frac{2}{1-3x} + \frac{1}{2x+1} - \frac{3}{(2x+1)^2}\right] dx \\
&= 2\int \frac{1}{1-3x}\,dx + \int \frac{1}{2x+1}\,dx - 3\int \frac{1}{(2x+1)^2}\,dx \\
&= -\frac{2}{3}\ln|1-3x| + \frac{1}{2}\ln|2x+1| - 3\left[\frac{(2x+1)^{-1}}{(-1)(2)}\right] + c \\
&= -\frac{2}{3}\ln|1-3x| + \frac{1}{2}\ln|2x+1| + \frac{3}{2(2x+1)} + c
\end{aligned}$$